

An integral in the complex plane: three weddings and a funeral

Let γ be the curve $z(t) = 2e^{it}$ for $0 \leq t \leq \pi$. Consider the integral

$$\int_{\gamma} \frac{1}{z^2 + 1} dz.$$

This is the integral I made a mess of in class, at least pedagogically. Namely I realized in time that I didn't yet have the technique, and I stopped.

This integral is an excellent exercise at a variety of levels. Let us start with the funeral. Please examine Method One to see what is incorrect.

Method One: Easy. Wrong.

$$\begin{aligned} \int_{\gamma} \frac{1}{z^2 + 1} dz &= \int_0^{\pi} \frac{1}{z(t)^2 + 1} z'(t) dt = \int_0^{\pi} \frac{2ie^{it}}{4(e^{it})^2 + 1} dt \stackrel{\boxed{u = e^{it}}}{=} 2 \int_1^{-1} \frac{du}{4u^2 + 1} \\ &\stackrel{\boxed{v = 2u}}{=} \int_2^{-2} \frac{dv}{v^2 + 1} = -2 \int_0^2 \frac{dv}{v^2 + 1} = -2 \arctan 2 \approx -2.2143. \end{aligned}$$

Method Two: Direct, by real means. Spectacular (i.e. ugly).

$$\begin{aligned} \int_{\gamma} \frac{1}{z^2 + 1} dz &= \int_0^{\pi} \frac{2ie^{it}}{4e^{2it} + 1} dt = 2i \int_0^{\pi} \frac{\cos t + i \sin t}{4 \cos 2t + i4 \sin 2t + 1} dt \\ &= 2i \int_0^{\pi} \frac{(\cos t + i \sin t)(4 \cos 2t + 1 - i4 \sin 2t)}{(4 \cos 2t + 1)^2 + (4 \sin 2t)^2} dt \\ &= 2i \int_0^{\pi} \frac{4(\cos t \cos 2t + \sin t \sin 2t) + \cos t + [4(\sin t \cos 2t - \cos t \sin 2t) + \sin t]}{16 \cos^2 t + 16 \sin^2 t + 8 \cos 2t + 1} dt \\ &= 2i \int_0^{\pi} \frac{4 \cos(t - 2t) + \cos t + [4 \sin(t - 2t) + \sin t]}{16 + 8 \cos 2t + 1} dt \\ &= 2i \int_0^{\pi} \frac{4 \cos t + \cos t + [-4 \sin t + \sin t]}{16 + 8 \cos 2t + 1} dt \\ &= 2i \int_0^{\pi} \frac{5 \cos t - 3i \sin t}{17 + 8 \cos 2t} dt \\ &= 2i \left(5 \int_0^{\pi} \frac{\cos t}{17 + 8(1 - 2 \sin^2 t)} dt - 3i \int_0^{\pi} \frac{\sin t}{17 + 8(2 \cos^2 t - 1)} dt \right) \\ &= 2i \left(5 \int_0^{\pi} \frac{\cos t}{25 - 16 \sin^2 t} dt - 3i \int_0^{\pi} \frac{\sin t}{9 + 16 \cos^2 t} dt \right) \\ &= 2i \left(5 \int_0^0 \frac{du}{25 - 16u^2} - 3i \int_1^{-1} \frac{-dv}{9 + 16v^2} \right) \end{aligned}$$

$$\begin{aligned}
&= 6 \int_{-1}^1 \frac{dv}{9 + 16v^2} = 6 \int_{-4/3}^{4/3} \frac{(3/4)dw}{9 + 9w^2} \\
&= \int_0^{4/3} \frac{dw}{1 + w^2} = \arctan(4/3) \approx 0.927295218
\end{aligned}$$

The techniques used here are all familiar to you. The equalities above are a twelve step program:

- i) use the definition,
- ii) Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$,
- iii) make denominator positive by multiplying top and bottom by conjugate ... ,
- iv) expand and collect,
- v) trig identities
- vi) cos is even and sin is odd,
- vii) astonishment that "17" appears,
- viii) set up for substitution (trig identities)
- ix) set up for substitution (trig identities)
- x) do substitution $u = \sin t$, $v = \cos t$
- xi) another substitution $w = (4/3)v$
- xii) finish up

Method Three: Theory we will get to ("residues"). Sweet.

Let $\tilde{\gamma}$ be the closed contour which is formed from γ above and the curve $z(t) = t$, $-2 \leq t \leq 2$. This closed contour is pictured at right. Note that the contour goes around i but not $-i$.

We will see later, in section 63 on Cauchy's residue theorem, that if we rewrite

$$\frac{1}{z^2 - 1} = \frac{\phi(z)}{z - i}$$

where $\phi(z) = 1/(z + i)$, then

$$\text{Res}_{z=i} = \phi(i) = \frac{1}{2i}.$$

On the other hand, Cauchy's residue theorem will say

$$\int_{\tilde{\gamma}} \frac{1}{z^2 - 1} dz = 2\pi i \text{Res}_{z=i} = \pi.$$

But then

$$\int_{\tilde{\gamma}} \frac{1}{z^2 - 1} dz = \int_{\tilde{\gamma}} \frac{1}{z^2 - 1} dz - \int_{-2}^2 \frac{1}{x^2 - 1} dx = \pi - 2 \int_0^2 \frac{1}{x^2 - 1} dx = \pi - 2 \arctan 2 \approx 0.927295218$$

Relative to Method One, we get a shift by π in this answer. That's good.

Note that this gets us the right answer and even "explains" what went wrong with Method One, as long as we firmly believe (i.e. we have a proof of) Cauchy's residue theorem.

Method Four: Numerical analysis (to get close ...). Robust. The above may have made you insecure. When this happens to me, my inclination is to approximate integrals by sums. Also to visualize integrands before integrating them. In this case I returned to the concrete integral which was our starting point,

$$\int_0^{\pi} \frac{2ie^{it}}{4e^{2it} + 1} dt.$$

Here is some MATLAB to compute the integral approximately by the trapezoid rule.¹:

```
% generate the parameter and the function to integrate (w(t)):
t = 0:pi/1000:pi;
u = exp(i*t);
w = 2*i*u ./ (4*u.^2 + 1);

% visualize real and imaginary parts of integrand:
plot(t,real(w),t,imag(w)), grid on

% setup coefficients for trapezoid rule; do it:
c = 2*ones(1,1001); c(1) = 1; c(1001)=1;
(pi/1000) * 0.5 * sum(c * w')
```

The result I get from this script is

```
ans =    0.927294823217621 + 2.32379238679825e-17i
```

and a picture below of the real and imaginary parts of the integrand.

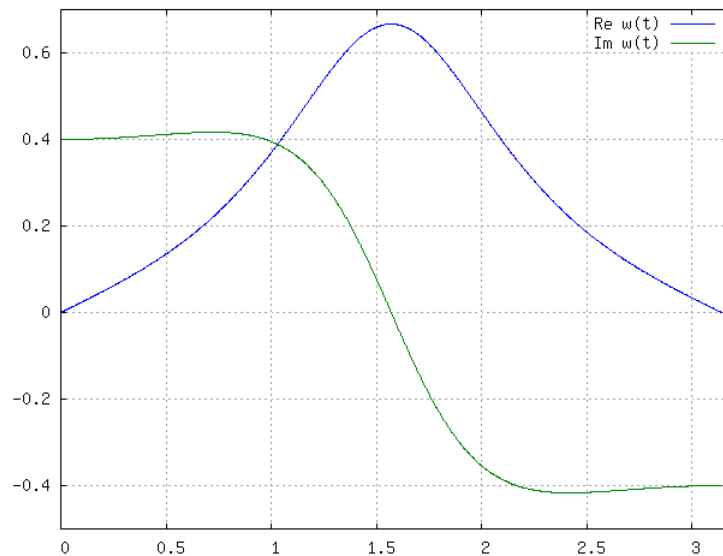


FIGURE 1. Real and imaginary parts of $w(t) = \frac{2ie^{it}}{4e^{2it} + 1}$.

¹Actually, I wrote this in OCTAVE (<http://www.gnu.org/software/octave/>), a free but incomplete alternative; it should run in MATLAB with no modification.