Solutions to Take-home Final Exam

Lesson 24, #7. Solution. The application of Leibniz rule is slightly simplified by having the integrand $\phi(s)$ not depend on the variable t, with respect to which we are differentiating. Thus with $\partial F/\partial t = 0$ and $f(t) = x - ct$ and $g(t) = x + ct$, we get

$$\frac{d}{dt}\left(\frac{1}{2c}\int_{x-ct}^{x+ct}\phi(s)\,ds\right) = \frac{1}{2c}\Big(0 + c\phi(x+ct) - (-c)\phi(x-ct)\Big) = \frac{1}{2}\phi(x+ct) + \frac{1}{2}\phi(x-ct).$$

This calculation is part of showing that D'Alembert's solution (17.8) giving $u(x,t)$, wherein ϕ is called g, actually solves the full wave equation problem including the initial condition $u_t(x,0) = g(x)$. In fact, the result above simplifies to $\phi(x)$ when we substitute $t = 0$.

Lesson 31, #2. Solution. This one is too easy:

$$u_{tt} = \alpha^2\left(u_{rr} + \frac{1}{r}u_r\right).$$

E1. (a). Solution. The separated solution is $u(x,t) = X(x)T(t)$. The parts solve ODEs

$$X'' + \lambda^2 X = 0, \qquad T' = -\lambda^2 T.$$

The eigenproblem is

$$X'' + \lambda^2 X = 0, \qquad X'(0) = 0, X'(L) = 0$$

which has solutions $\lambda_n = n\pi/L$, $X_n(x) = \cos(n\pi x/L)$, $n = 1, 2, 3, \dots$. The associated time-dependent solutions are $T_n(t) = A_n e^{-n^2\pi^2 t/L^2}$. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t/L^2}\cos(n\pi x/L).$$

The initial condition requires

$$e^{-x} = u(x,0) = \sum_{n=1}^{\infty} A_n \cos(n\pi x/L)$$

for $0 < x < L$.

Line 9 in Table E in the back of the book says that on $0 < y < \pi$ we have

$$e^{ay} = \frac{2a}{\pi}\left[\frac{e^{a\pi} - 1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n e^{a\pi} - 1}{n^2 + a^2}\cos(ny)\right].$$

Substituting $y = \pi x/L$ for $0 < x < L$ gives

$$e^{a\pi x/L} = \frac{2a}{\pi}\left[\frac{e^{a\pi} - 1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n e^{a\pi} - 1}{n^2 + a^2}\cos(\pi n x/L)\right].$$

Now let $a\pi/L = -1$ or $a = -L/\pi$ to get

$$\begin{aligned}
e^{-x} &= \frac{-2L}{\pi^2}\left[\frac{e^{-L} - 1}{2L^2/\pi^2} + \sum_{n=1}^{\infty}\frac{(-1)^n e^{-L} - 1}{n^2 + (L/\pi)^2}\cos(\pi n x/L)\right] \\
&= -2L\left[\frac{e^{-L} - 1}{2L^2} + \sum_{n=1}^{\infty}\frac{(-1)^n e^{-L} - 1}{\pi^2 n^2 + L^2}\cos(\pi n x/L)\right] \\
&= \frac{1 - e^{-L}}{L} + 2L\sum_{n=1}^{\infty}\frac{1 - (-1)^n e^{-L}}{\pi^2 n^2 + L^2}\cos(\pi n x/L)
\end{aligned}$$

on $0 < x < L$ as desired. Thus

$$u(x,t) = \frac{1-e^{-L}}{L} + 2L \sum_{n=1}^{\infty} \frac{1-(-1)^n e^{-L}}{\pi^2 n^2 + L^2} e^{-n^2\pi^2 t/L^2} \cos(\pi n x/L).$$

(b). Solution. Let $U(\xi,t) = \mathcal{F}[u]$. Then by properties of Fourier transform

$$U_t(\xi,t) = -\xi^2 U(\xi,t).$$

and $U(\xi,0) = \Phi(\xi) = \mathcal{F}[\phi]$. This is an ODE IVP for each ξ.

Recalling $\phi(x) = e{-}|x|$, by line 7 of Table A,

$$\Phi(\xi) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\xi^2}.$$

Therefore the solution of these ODE IVPs is

$$U(\xi,t) = \sqrt{\frac{2}{\pi}} \frac{e^{-\xi^2 t}}{1+\xi^2}.$$

The inverse Fourier transform we want now is the integral

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{e^{-\xi^2 t}}{1+\xi^2} e^{ix\xi}\, d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-\xi^2 t + ix\xi}}{1+\xi^2}\, d\xi.$$

But this problem seems to be too hard to do directly.

On the other hand, the book gives formula (12.9), which in our case is

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-|y|} e^{-(x-y)^2/(4t)}\, dy.$$

Consider the integral in two halves and use the hint distributed by email:

$$\int_0^{\infty} e^{-y} e^{-(x-y)^2/(4t)}\, dy = \sqrt{\pi t}\, e^{t-x} \operatorname{erfc}\left(-\frac{x}{2\sqrt{t}} + \sqrt{t}\right),$$
$$\int_{-\infty}^{0} e^{+y} e^{-(x-y)^2/(4t)}\, dy = \sqrt{\pi t}\, e^{t+x} \operatorname{erfc}\left(+\frac{x}{2\sqrt{t}} + \sqrt{t}\right).$$

Thus

$$u(x,t) = \frac{1}{2} e^{t+x} \operatorname{erfc}\left(+\frac{x}{2\sqrt{t}} + \sqrt{t}\right) + \frac{1}{2} e^{t-x} \operatorname{erfc}\left(-\frac{x}{2\sqrt{t}} + \sqrt{t}\right).$$

(c). Solution. There are two key ideas here:

- On the interval $[0,L]$ the initial conditions are the same, namely $u(x,0) = e^{-x}$, and therefore the solutions start with the "same first frame" if we restrict our view to $[0,L]$.
- The boundary conditions in part **(a)** are insulating while the rod is infinitely long in part **(b)**. Thus over a long time we expect the solution in part **(a)** to approach a nonzero constant temperature, while in part **(b)** we expect the temperature in the rod to go to zero everywhere.

I have posted a code that shows movies of the solutions to parts **(a)** and **(b)**,

`http://www.dms.uaf.edu/~bueler/compareheat.m`

Figure 1 shows three frames from that movie.

E2. (a). Solution. Applying $\mathcal{F}$ to the PDE we get

$$U_{tt}(\xi,t) = -9\xi^2 U(\xi,t).$$

The solutions to these ODEs are

$$U(\xi,t) = A(\xi)\cos(3\xi t) + B(\xi)\sin(3\xi t).$$

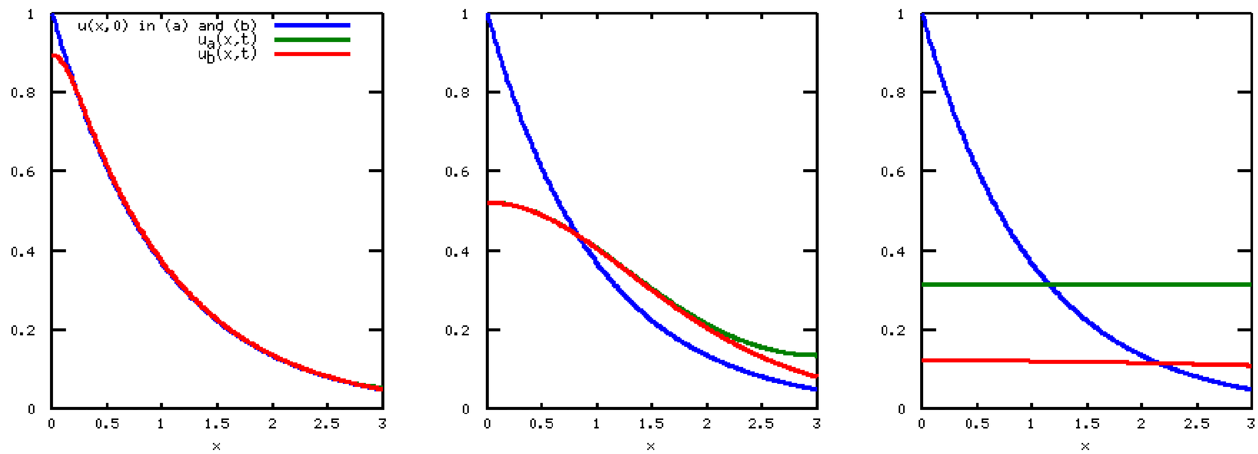

FIGURE 1. Solutions $u(x,t)$ from parts **(a)** and **(b)** of E1 at times $t = 0.01, 0.5, 20$ in the case where $L = 3$. Note that both parts agree with e^{-x} on the interval $[0, L]$ at the smallest time. As time goes on, the fact that the boundary conditions in part **(a)** are insulating is revealed.

But the initial values are $U(\xi, 0) = \Phi(\xi)$ and $U_t(\xi, 0) = 0$. Thus $A(\xi) = \Phi(\xi)$ and $B(\xi) = 0$. Therefore

$$U(\xi, t) = \Phi(\xi)\cos(3\xi t).$$

(b). Solution. Easy:

$$u(x,t) = \frac{1}{2}\phi(x - 3t) + \frac{1}{2}\phi(x + 3t).$$

E3. (a). Solution. The equation is linear but variable coefficient. If $\lambda = 0$ then the equation is

$$x^2 y'' + xy' = 0$$

which simplifies to

$$xw' + w = 0$$

with $w = y'$ as suggested. This is separable,

$$x\frac{dw}{dx} + w = 0 \qquad \iff \qquad \frac{dw}{w} = -\frac{dx}{x}.$$

Integrating gives

$$\ln|w(x)| = -\ln|x| + C.$$

Exponentiating this and recalling we are only considering $x > 0$ gives

$$y'(x) = w(x) = Ae^{-\ln x} = \frac{A}{x}$$

for A which is any nonzero real number. Integrating this gives

$$y(x) = A\ln x + B.$$

This is the general solution.

(b). Solution. Now suppose $\lambda > 0$ and substitute x^r for y, to get

$$x^2 r(r-1)x^{r-2} + xrx^{r-1} - \lambda^2 x^r = 0.$$

This simplifies to

$$r^2 - \lambda^2 = 0$$

with solutions $r = \pm\lambda$. That is, the general solution

$$y(x) = Ax^{\lambda} + Bx^{-\lambda}.$$

(c). Solution. In Lesson 33 we seek the separated solutions $u(r,\theta) = R(r)\Theta(\theta)$ of the Laplacian equation $\nabla^2 u = 0$ on the unit disc. The factor $R(r)$ solves the Euler equation

$$r^2 R'' + rR' - \lambda^2 R = 0$$

where $\lambda = n$ is the eigenvalue; note $\Theta(\theta)$ is periodic and solves $\Theta'' + \lambda^2\Theta = 0$ so $\lambda = n$. The solutions $\ln r$ and r^{-n} are, however, unbounded at the center of the disc, $r \to 0$. Thus we throw those away and keep the solutions

$$R(r) = \begin{cases} A, & n = 0, \\ Ar^n, & n > 0, \end{cases}$$

but this formula simplifies to $R(r) = Ar^n$ in all cases $n = 0, 1, 2, 3, \ldots$.

E4. (a). Solution. If $u(x,t) = X(x)T(t)$ then the PDE says

$$\frac{T'' + \gamma T'}{\alpha^2 T} = -\lambda^2 = \frac{X''}{X}$$

so the factors solve

$$T'' + \gamma T' + \alpha^2\lambda^2 T = 0, \qquad X'' + \lambda^2 X = 0$$

for λ unknown.

(b). Solution. The eigenvalue problem is

$$X'' + \lambda^2 X = 0, \qquad X(0) = 0, \qquad X(1) = 0.$$

Thus

$$\lambda_n = n\pi, \qquad X_n(x) = \sin(n\pi x), \qquad n = 1, 2, 3, \ldots$$

(c). Solution. We now return to the equation for $T(t)$ and solve this constant-coefficient ODE,

$$T_n(t) = e^{zt} \qquad \Longrightarrow \qquad z^2 + \gamma z + \alpha^2 n^2\pi^2 = 0.$$

Thus

$$z = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\alpha^2 n^2\pi^2}}{2}$$

where $n = 1, 2, 3, \ldots$

But what is the sign of "$\gamma^2 - 4\alpha^2 n^2\pi^2$" in the square root? We are told to assume $\gamma < \alpha\pi$ so $\gamma^2 - \alpha^2\pi^2 < 0$. Thus $\gamma^2 - 4\alpha^2 n^2\pi^2 < 0$ for $n \geq 1$. Therefore we may write

$$z = -\frac{\gamma}{2} \pm i\frac{\sqrt{4\alpha^2 n^2\pi^2 - \gamma^2}}{2}.$$

Let

$$\zeta_n = \frac{\sqrt{4\alpha^2 n^2\pi^2 - \gamma^2}}{2}.$$

Then

$$T_n(t) = e^{-\gamma t/2}\left(a_n \cos(\zeta_n t) + b_n \sin(\zeta_n t)\right).$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\gamma t/2}\left(a_n \cos(\zeta_n t) + b_n \sin(\zeta_n t)\right)\sin(n\pi x).$$

We want

$$\sin(\pi x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

and

$$0 = u_t(x,0) = -\frac{\gamma}{2}\sin(\pi x) + \sum_{n=1}^{\infty} b_n \zeta_n \sin(n\pi x).$$

By orthogonality, $a_1 = 1$ and $a_n = 0$ for $n \ge 2$, and $b_1 = \gamma/(2\zeta_1)$ while $b_n = 0$ for $n \ge 2$. Thus

$$u(x,t) = e^{-\gamma t/2}\left(\cos(\zeta_1 t) + \frac{\gamma}{2\zeta_1}\sin(\zeta_1 t)\right)\sin(\pi x)$$

where $\zeta_1 = \sqrt{4\alpha^2\pi^2 - \gamma^2}/2$. *Yes, you can check this solves all parts of the problem!*

E5. (a). Solution. Because $f(x) = |x|$ is even, a Fourier cosine series is needed. From Table E in the back of the book,

$$|x| = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)^2}\cos((2k+1)x).$$

Thus the discrete spectrum has

$$c_0 = \frac{\pi}{2}, \qquad c_{2l} = 0, \qquad c_{2k+1} = \frac{4}{\pi(2k+1)^2},$$

where $l = 1, 2, 3, \dots$ and $k = 0, 1, 2, 3, \dots$

(b). Solution. For this function we compute the classical Fourier series

$$e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin(nx)\,dx.$$

Clearly we must do integrations by parts, giving:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos(nx)\,dx = \frac{(-1)^n(e^\pi - e^{-\pi})}{\pi(n^2+1)}, \qquad n = 0, 1, 2, 3, \dots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin(nx)\,dx = \frac{-(-1)^n n(e^\pi - e^{-\pi})}{\pi(n^2+1)}, \qquad n = 1, 2, 3, \dots \end{aligned}$$

Thus

$$c_0 = \frac{e^\pi - e^{-\pi}}{\pi} \qquad \text{and} \qquad c_n = \frac{(e^\pi - e^{-\pi})}{\pi\sqrt{n^2+1}}, \quad n = 1, 2, 3, \dots$$

The figures for the two parts are easy to draw and therefore omitted. Note that in the first case the coefficients decay like $O(n^{-2})$ as $n \to \infty$, because $f(x)$ has a continuous periodic extension, while in the second case the coefficients decay only like $O(n^{-1})$ because the periodic extension is discontinuous.

E6. Solution. We separate variables $u(x,y) = X(x)Y(y)$ and get this equation

$$X''Y + XY'' + \lambda^2 XY = 0,$$

which splits as

$$\frac{X'' + \lambda^2 X}{X} = \mu^2 = \frac{-Y''}{Y}$$

for some unknown constant μ. The boundary conditions give this eigenvalue problem for Y,

$$Y'' + \mu^2 Y = 0, \qquad Y(0) = 0, \qquad Y(1) = 0,$$

because $u = 0$ on the boundary of the square. This familiar eigenvalue problem has the familiar solution

$$Y_n(y) = \sin(n\pi y), \qquad \mu_n = n\pi, \qquad n = 1, 2, 3, \dots$$

Now we return to the ODE problem for $X(x)$ and solve it. Again it is an eigenvalue problem, this time for eigenvalue λ. The ODE is $X'' + \lambda^2 X = n^2\pi^2 X$ or equivalently $X'' + (\lambda^2 - n^2\pi^2)X = 0$. The full eigenproblem is

$$X'' + (\lambda^2 - n^2\pi^2)X = 0, \qquad X(0) = 0, \qquad X(1) = 0,$$

again because $u = 0$ on the boundary of the square. We have

$$X_m(x) = \sin(m\pi x)$$

for $m = 1, 2, 3, \dots$ and where $\lambda^2 - n^2\pi^2 = m^2\pi^2$. Thus

$$\lambda = \lambda_{m,n} = \pi\sqrt{m^2 + n^2}$$

over all pairs (m, n) with $m = 1, 2, 3, \dots$ and $n = 1, 2, 3, \dots$ A bit of thought shows that this last formula for λ must treat m and n symmetrically, because the original problem treated x and y symmetrically.

The solutions to the original eigenfunction problem are

$$u_{m,n}(x, y) = \sin(m\pi x)\sin(n\pi x)$$

with $\lambda = \lambda_{m,n}$ already given.

Extra Credit 1. Solution. I wrote a program which is posted online:

`http://www.dms.uaf.edu/~bueler/squaredrums.m`

It generates Figure 2.

Extra Credit 2. Solution. *No*, you cannot hear the shape of a drum. Google "can you hear the shape of a drum?"; the ease of doing this is why this is a *one*-credit extra credit problem.

Note that the problem solved in Lecture 30, and the problem solved above in **E6**, give different sequences of eigenvalues. These eigenvalues are the tones generated by the drumheads, one round and one square. Any sound from these drums is a linear combination of these frequencies. Yes, you can hear the difference between a round drum and a square drum, because of the differences in these sequences. You can hear the difference between a round drum and a square drum even if they have the same area. Generally, also, you can hear the difference in the boundary length of drums if they have the same area.

But what if the drums have the same area and boundary length? Could two such drums have actually different shapes but sound the same. This was an open mathematical question from the time of the publication of this paper,

> Kac, Mark (1966), *Can one hear the shape of a drum?*, American Mathematical Monthly **73** (4, part 2) 1–23.

My guess is that most knowledgable mathematicians in the 1960s and 1970s would say "someday it will be proven that you *can* hear the shape of a 2-dimensional drumhead". But the problem was solved negatively, as noted, in this paper

> Gordon, C., Webb, D. L., and Wolpert, S. (1992), *One cannot hear the shape of a drum*, Bulletin of the American Mathematical Society **27** (1), 134–138

Here are two drum heads that sound the same, that is, they have *all* of the same eigenvalues λ_n solving the Helmholtz-Dirichlet problem, namely $\nabla^2 u + \lambda^2 u = 0$, where u is zero on the boundary of the drumhead (rim of the drum):

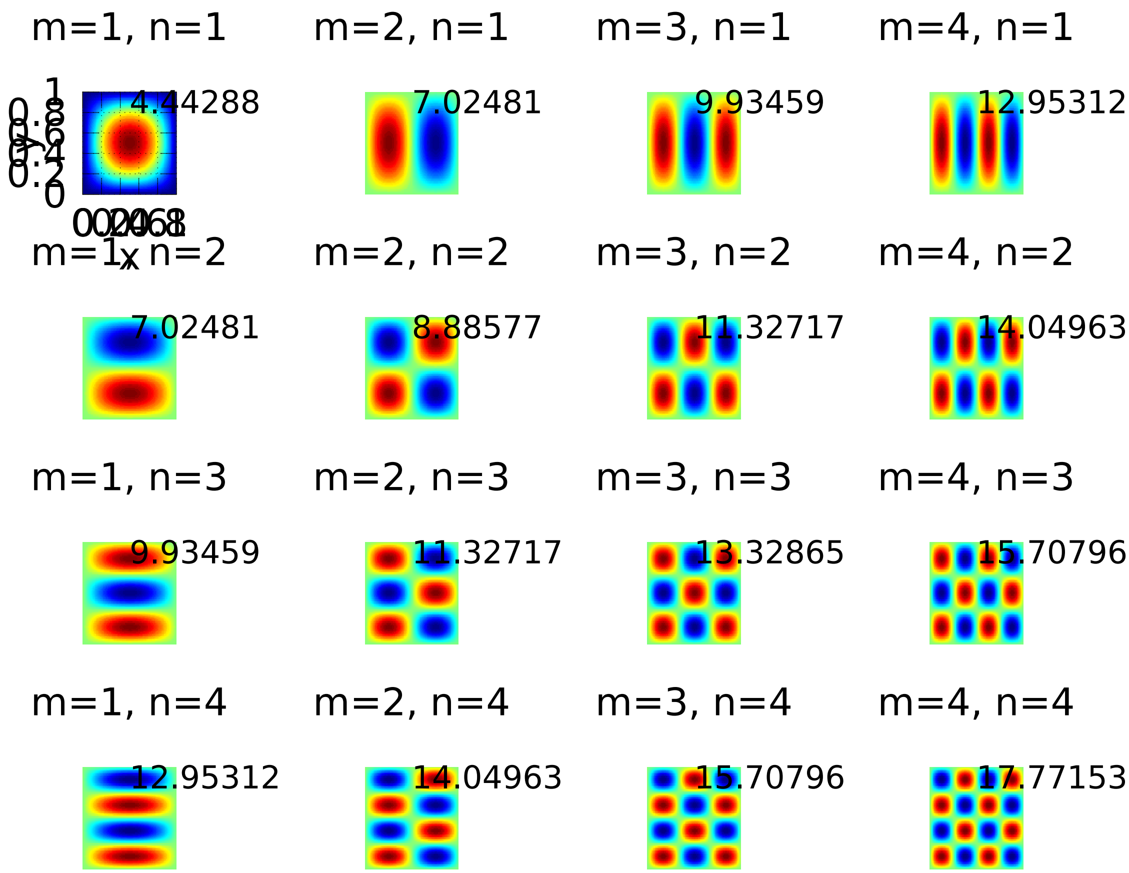

FIGURE 2. The first sixteen eigenfunctions for the Laplacian on a unit square, with their associated values of $\lambda_{m,n} = \pi\sqrt{m^2 + n^2}$. These are $u_{m,n}(x, y)$ for $m = 1, 2, 3, 4$ and $n = 1, 2, 3, 4$ from the solution to **E6**.

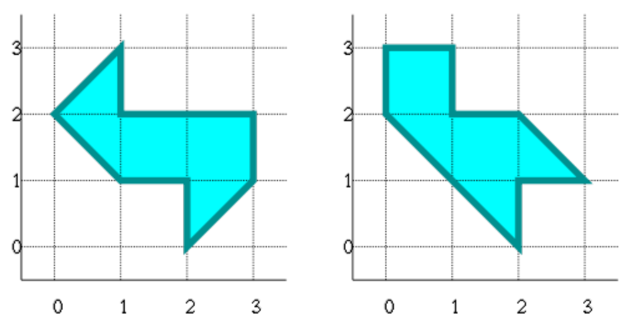

FIGURE 3. Two drumheads that sound exactly the same.