Take-home Final Exam

100 points total

Due Thursday 15 December, 2011 at 5pm.

Rules. You *may* use written and published references for basic facts including trigonometric identities and integrals, but you should indicate clearly when doing so; *cite*! You *may* use the textbook. You *may* use MATLAB and other calculation technology to produce plots and compute numbers. You *may* ask questions during class time about the content of the exam, so that you and all the other students hear the question and the response. You *may* talk to me in person or email me with questions at `elbueler@alaska.edu`.

You *may not* seek out complete solutions by searching online, in textbooks, or elsewhere. You *may not* talk to, or communicate with by any method, any person other than *me* about the content of this exam.

Please come to me with questions, including questions about these rules. I will be genuinely helpful and not obscure; I'll tell you when I have stopped revealing information so as to avoid giving an answer.

Lesson 24, # 7. *(5 points)*

Lesson 31, # 2. *(5 points)*

E1. (a) *(10 points)* Use separation of variables to solve this heat equation problem on $0 \le x \le L$:

PDE	$u_t = u_{xx}$
BCs	$u_x(0,t) = 0$ $u_x(L,t) = 0$
IC	$u(x,0) = e^{-x}$

Show all the major steps, even if they appear in the textbook.

(b) *(10 points)* Use Fourier transforms to solve this initial value problem for the heat equation on $-\infty < x < \infty$:

PDE	$u_t = u_{xx}$
IC	$u(x,0) = e^{-\lvert x \rvert}$

Show all the major steps, even if they appear in the textbook. (*Hint*: You may use a table of Fourier transforms.)

(c) *(5 points)* Assume $L = 2$ for concreteness. Compare the solutions above by plotting them both on the interval $0 \le x \le L$. In particular, for both part **(a)** and part **(b)** sketch the solutions $u(x,t)$ at $t = 0$, at a small time $t > 0$, and at a large time $t > 0$. *Clearly* label. Expect to lose points if your plots are hard to understand, vague, or messy. (*Hint*: Pay attention to what boundary conditions apply and what are the effects of these. Also, what survives for a long time?)

E2. **(a)** *(5 points)* Consider the initial value problem for the wave equation on $-\infty < x < \infty$:

PDE	$u_{tt} = 9u_{xx}$
ICs	$u(x,0) = \phi(x)$ $u_t(x,0) = 0$

where $\phi(x)$ is some function for which $\int_{-\infty}^{\infty} |\phi(x)|^2\,dx < \infty$, and where $u = u(x,t)$ (as usual). We can apply the Fourier transform to this problem. In fact, define

$$U(\xi,t) = \mathcal{F}[u], \qquad \Phi(\xi) = \mathcal{F}[\phi],$$

and show that

$$U(\xi,t) = \Phi(\xi)\cos(3\xi t).$$

Show all significant steps of the calculation. (*Hint*: To start, just apply $\mathcal{F}$ to both sides of the PDE, and follow your nose. *Note*: The reason we do not proceed from the form $U(\xi,t) = \Phi(\xi)\cos(3\xi t)$ to a complete solution is that the inverse Fourier transform $\mathcal{F}^{-1}[\cos(3\xi t)]$ is not defined, using the tools presented in this class and textbook.)

(b) *(5 points)* What is D'Alembert's solution to the problem in part **(a)**?

E3. Consider *Euler's equation*

$$x^2 y'' + xy' - \lambda^2 y = 0,$$

which is an ODE for $y(x)$. We will only consider $x > 0$ here.

(a) *(5 points)* Is this equation linear? Is it constant-coefficient? Find the general solution in the case $\lambda = 0$. (*Hint*: The substitution $w(x) = y'(x)$ may be helpful.)

(b) *(5 points)* Find all solutions of the form $y(x) = x^r$ in the case $\lambda > 0$, and then find the general solution.

(c) *(5 points)* Write at least three clear, well-considered sentences describing how Euler's equation arises in **Lesson 33**, what role it plays in finding the solution, and which solutions are kept, and why.

E4. **(a)** *(5 points)* Apply separation of variables to this PDE, a transmission line equation (see **Lesson 16**):

$$\text{PDE} \qquad u_{tt} + \gamma u_t = \alpha^2 u_{xx}.$$

In particular, write down the ODEs solved by each factor of the separated form of the solution $u(x,t)$. Assume in this part, and below, that $\gamma \geq 0$ and $\alpha > 0$.

(b) *(5 points)* State and fully solve the eigenvalue problem associated to this PDE initial/boundary value problem:

$$\begin{array}{ll} \text{PDE} & u_{tt} + \gamma u_t = \alpha^2 u_{xx} \\ \text{BCs} & \begin{array}{l} u(0,t) = 0 \\ u(1,t) = 0 \end{array} \\ \text{IC}s & \begin{array}{c} u(x,0) = \sin(\pi x) \\ u_t(x,0) = 0 \end{array} \end{array}$$

(c) *(10 points)* Solve the initial/boundary value problem stated in part **(b)**. You may assume, at any point you need this fact, that $\gamma < \pi\alpha$, so γ is not big.

E5. Graph the discrete frequency spectrum of these periodic functions. Put at least 10 reasonably-accurate points on each spectrum graph. Regard these functions as having the given formula on $(-\pi, \pi)$, and suppose they repeat periodically with period 2π. (*Hint*: "Discrete frequency spectrum" is defined in **Lesson 11**.)

(a) *(5 points)* $f(x) = |x|$

(b) *(5 points)* $f(x) = e^x$

E6. *(10 points)* Consider the Laplacian on the square $0 < x < 1$, $0 < y < 1$, namely $\nabla^2 u = u_{xx} + u_{yy}$. Use separation of variables to solve the eigenfunction problem

$$\nabla^2 u + \lambda^2 u = 0$$

assuming the value of u on the boundary of the square is always zero.

(Extra Credit 1) *(3 points)* Write a computer program that shows, for this square, the drumheads analogous to the disc case shown in Figure 30.3 in **Lesson 30**. In particular, show at least 16 drumhead modes.

(Extra Credit 2) *(1 points)* (*For this problem you may, and surely you must, search online for information.*) Can you hear the shape of a drum? Draw two drumheads which answer this question.