Final Exam

Due May 8 5:15 pm (end of final period) *FIRM.*

1. Easier (50 points)

I.1 (*5 points*) **Exercise #6 in section 1–3.**

I.2 (*5 points*) **Exercise #2 in section 2–2.** [*Carefully use the definition; not hard!*]

I.3 (*5 points*) **Exercise #4 in section 1–7.** [*This is a good warmup because an outline of the answer appears in the back—you must still fill in the details!*]

I.4 (*5 points*) **Exercise #5 in section 1–7.** [*Again, fill in the details!*]

I.5 (*5 points*) **Exercise #9a in section 3–2.** [*Work from properties of differentials of maps.*]

I.6 (*10 points*) **a.** Any helix $\alpha(t) = (\cos at, \sin at, bt)$, $a^2 + b^2 = 1$, is an arclength–parameterized curve in the surface $S = \{x^2 + y^2 = 1\}$. The points $p = (1, 0, 0) = \alpha(0)$ and $q = (-1, 0, 1)$ are distinct points on S. Find *all* values of a, b which correspond to helices connecting p to q. (There is a countable list of such helices $\{\alpha_n\}$.) [*This problem makes the most sense if you read pages 246, 247.*]

b. Find a local isometry $\varphi : \mathbf{R}^2 \to S$ so that the preimage of each of the curves α_n answering **a** is a straight line in $\mathbf{R}^2$.

c. For each of the answers α_n in **a**, calculate the normal curvature $k_n(p)$. Show $\lim_n k_n(p)$ is a principal curvature of S at p.

I.7 (*10 points*) **Exercise #2 in section 4–2.**

I.8 (*5 points*) *True* or *False*: Is it true that if $\varphi : S_1 \to S_2$ is a local isometry and if the mean curvature $H_1 = 0$ at all points of S_1 then $H_2 = 0$ at all points of S_2? If true give proof. If false, give counterexample (that is, a pair of surfaces and a local isometry).

2. Longer calculation or more abstract (50 points)

II.1 (*10 points*) **Exercise #11 in section 2–5.** [*Use the methods of 2–5, that is, start from the definition of area given on page 98.*]

II.2 (*10 points*) **Exercise #17 in section 3–2.** [*Note type: "... intersecting curves on* S *and ..."*]

II.3 (*10 points*) **a.** Graph the surface S described by

$$\mathbf{x}(u, v) = (au \cos v, bu \sin v, u^2)$$

for $a, b > 0$ and $0 < u < \infty$, $0 < v < 2\pi$. [*Describe the boundaries of the surface carefully, as well.*]

b. Compute the Gauss curvature K on S. Redo the calculation—much easier!—if $a = b$.

II.4 (*10 points*) Compute E, F, G, e, f, g for the unit sphere in spherical coordinates: $\mathbf{x}(u, v) = (\cos v \cos u, \cos v \sin u, \sin v)$, $-\pi/2 < v < \pi/2$, $0 < u < 2\pi$. [*You can check your calculation by using your knowledge of curvatures K, H or the principal curvatures.*]

II.5 (*10 points*) Use the Gauss–Bonnet theorem to show that if S is a compact, connected, oriented surface strictly containing the unit sphere then there exists a point p on S such that $K(p) < 1$. [*You may assume two crucial facts. First that "containing" means exactly what you think it does, even if it is hard to define. Second that the unit sphere is the surface of smallest area containing the ball $\{x^2+y^2+z^2 < 1\}$, and the only such surface with area 4π. Furthermore you may use any proposition given in 4–5.*]

Rules for this Take–Home Exam

1. I (Ed Bueler) am the only person with whom you may communicate regarding this exam.
2. You may use any *book* you wish, but do give references (for books other than the textbook).
3. Do not put more than one problem per page (i.e. at most two problems per sheet if you use both sides).
4. I will deduct points for unreadable messes!
5. You may work on this final outside.
6. When you turn in your exam, sort your solutions into the order given and staple in the upper left corner.