Math 404 Topology (Bueler) 7 November, 2016

Sort the alphabet into homeomorphism classes

1. Sans serif

Consider the alphabet written in sans serif font[1] as below. Suppose we accept an idealized notion of the letters as *closed subsets of the plane* $\mathbb{R}^2$ *which are built from finitely-many smooth curves*. Note that smooth curves have zero width, and that the interiors of the letters are empty by this definition. For example, the letter "I" is a single closed line segment, the letters "LTV" can all be built from two closed line segments, the letters "CJU" require single smooth curves, and so on.

A B C D E F G H I

J K L M N O P Q R

S T U V W X Y Z

Your job is to sort the above alphabet into homeomorphism classes, thinking of each letter as a topological space. Each letter has the subspace topology induced by the usual topology on $\mathbb{R}^2$. Recall that equivalence of topological spaces, i.e. the existence of a homeomorphism between them, is an equivalence relation. Thus you will partition the set of all letters into subsets of homeomorphic letters.

Some tools will help, as follows. In fact the goal is not to be totally rigorous, which would require writing down precise formulas for many homeomorphisms. The goal is, however, to give compelling arguments for why there are or are not homeomorphisms between pairs of letters. So, first observe that each letter is a connected topological space. Then apply these two theorems, which you do not need to prove right now:

Theorem 1. *Suppose* X, Y *are topological spaces,* $f : X \to Y$ *is a homeomorphism, and* $c \in X$*. Then* $\hat{f} : X \setminus \{c\} \to Y \setminus \{f(c)\}$ *is also a homeomorphism.*

Theorem 2. *Homeomorphic topological spaces have the same number of connected components.*

[1] A "serif" font has small extra strokes (serifs) at the ends of the major strokes of the letter, like ABC. A "sans serif" font is without serifs, like ABC.

2. Bold sans serif

Redo the exercise assuming that the strokes of the letters have positive thickness, as below.

A B C D E F G H I

J K L M N O P Q R

S T U V W X Y Z

In this case, for example, the letter **C** is homeomorphic to a round, filled disc ●. It would be hard to write down a formula for this homeomorphism but I believe it is easy to believe that there is one. That is, one can map points of the **C** to points of the ● by a continuous bijection in both directions.

Extra Credit. Assuming that the letter **I** is exactly a filled rectangle, write down a formula for a homeomorphism between **I** and the disc ●.

3. Serif

Redo the exercise for a serif font, but assuming as in part one that the strokes have zero width:

A B C D E F G H I

J K L M N O P Q R

S T U V W X Y Z

4. Lower case san serif

Redo the exercise assuming that the strokes have zero width:

a b c d e f g h i

j k l m n o p q r

s t u v w x y z