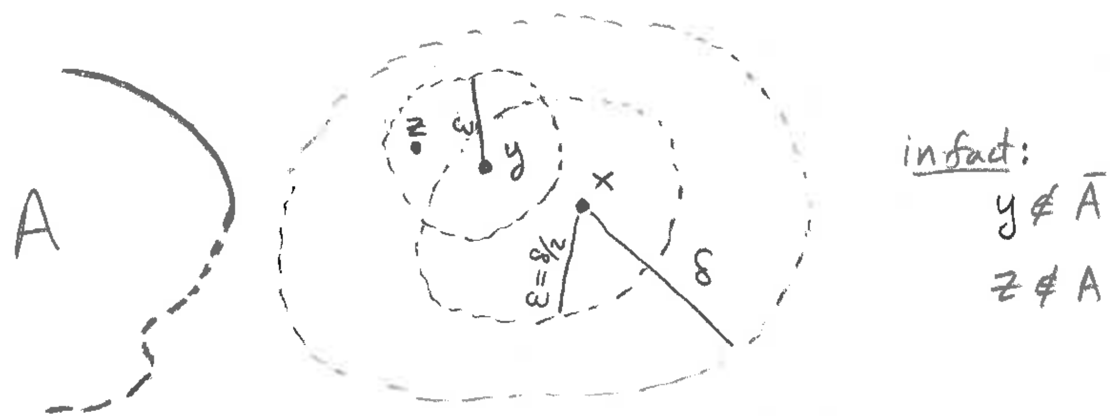
A difficult point about closures

Because I was unprepared, I walked into a trap on Monday. There is surprising subtlety about closures. You actually need a triangle inequality argument for some of the facts, including Proposition 6.11 (d) and (e) below. Unwisely, I asked you to prove these directly from the definitions.

In this note I do it. Use the picture below to understand the Lemma, which is the hard part. It is then easy to prove parts (d) and (e).[1] The corresponding facts about interiors are similar.



Lemma. *Suppose (X, d) is a metric space and $A \subseteq X$. If $x \in X \setminus \bar{A}$ then there is $\epsilon > 0$ so that $B_\epsilon(x) \cap \bar{A} = \emptyset$.*

Proof. Suppose $x \in X \setminus \bar{A}$. Since $x \notin \bar{A}$, by negating the definition of point of closure we know there is $\delta > 0$ so that $B_\delta(x) \cap A = \emptyset$. Let $\epsilon = \delta/2$.

We claim $B_\epsilon(x) \cap \bar{A} = \emptyset$. Otherwise there is $y \in B_\epsilon(x) \cap \bar{A}$. Because $y \in \bar{A}$ it follows that $B_\epsilon(y) \cap A \neq \emptyset$. Let $z \in B_\epsilon(y) \cap A$. Then

$$d(x, z) \leq d(x, y) + d(y, z) < \epsilon + \epsilon = \delta$$

so $z \in B_\delta(x)$. Since also $z \in A$ this contradicts the fact that $B_\delta(x) \cap A = \emptyset$. □

Proposition 6.11 (d). $\bar{\bar{A}} = \bar{A}$.

Proof. ($\supseteq$) If $x \in \bar{A}$ then for all $\epsilon > 0$ it follows that $x \in B_\epsilon(x) \cap \bar{A}$, so $B_\epsilon(x) \cap \bar{A} \neq \emptyset$. Thus $x \in \bar{\bar{A}}$.
($\subseteq$) We show that $x \in \bar{\bar{A}} \implies x \in \bar{A}$ by showing the contrapositive, namely that $x \notin \bar{A} \implies x \notin \bar{\bar{A}}$. So suppose $x \notin \bar{A}$, which is to say $x \in X \setminus \bar{A}$. By the Lemma there is $\epsilon > 0$ so that $B_\epsilon(x) \cap \bar{A} = \emptyset$. By the definition of $\bar{\bar{A}}$, namely the closure of $\bar{A}$, we know $x \notin \bar{\bar{A}}$. □

Proposition 6.11 (e). $\bar{A}$ is closed in X.

Proof. We show $X \setminus \bar{A}$ is open. Let $x \in X \setminus \bar{A}$. By the Lemma there is $\epsilon > 0$ so that $B_\epsilon(x) \cap \bar{A} = \emptyset$. This means $B_\epsilon(x) \subseteq X \setminus \bar{A}$. Since $x \in X \setminus \bar{A}$ was arbitrary, this means $X \setminus \bar{A}$ is open and thus $\bar{A}$ is closed. □

[1] The book does these arguments too, but in a different order. The textbook gives convoluted arguments for both parts (c) and (d) of Proposition 6.11, and invokes Exercise 5.6. The arguments here are not really more efficient, but they are different and illustrated.