Finite sets and maxima

On page 20 of the textbook *Elementary Analysis*, the author Ken Ross asserts that "Every finite nonempty subset of $\mathbb{R}$ has a maximum and a minimum." Ross made a pedagogical decision which I understand: skip the proof and hope the reader doesn't mind! But let's suppose you want a proof. This short note does that.

In lecture on Monday 16 September I also made a overly-extended point about the definition of finite sets. Here is that definition written down, so I can wrap that up:

Definition. *Let A be a set. We say A is a* finite set *if there exists a natural number $n \in \mathbb{N}$ and a function $f : A \to \{1, 2, 3, \dots, n\}$ which is one-to-one and onto (a bijection).*[1] *We say that $n = |A|$ or that n is the* cardinality *of A.*

If you know the definition of "finite set" then a proof by induction of Ross' assertion is not surprising.

Theorem. *Every finite subset of $\mathbb{R}$ has a maximum.*

Proof. Let P_n be the proposition "if $S \subset \mathbb{R}$ is a finite set with cardinality n then S has a maximum". Proposition P_1 is true because the only element of S is obviously the maximum.[2]

Suppose P_n is true, that is, suppose that every finite subset $S \subset \mathbb{R}$ with cardinality n has a maximum. Suppose $T \subset \mathbb{R}$ is finite and it has cardinality $n + 1$. Thus there is $f : T \to \{1, 2, \dots, n, n + 1\}$ which is a bijection. Let $S = f^{-1}(\{1, 2, \dots, n\})$. Then, because f is a bijection, $T = S \bigcup \{y\}$ where $y = f^{-1}(n + 1)$.

Let $x \in S$ be the maximum of S, which exists by P_n. If $y \le x$ then x is greater than or equal to every element of T and so x, which is of course in T, is the maximum. Otherwise $y > z$ for every $z \in S$ so $y \in T$ is the maximum. □

The proof of the next theorem is the same except for exchanging "minimum" for "maximum" in every case where it occurs in the proof, changing the eighth sentence of the proof to

> If $y \ge x$ then x is less than or equal to every element of T and so x, which is of course in T, is the minimum.

and reversing the other inequality in the proof.

Theorem. *Every finite subset of $\mathbb{R}$ has a minimum.*

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[1] Notice that "$\{1, 2, 3, \dots, n\}$" might have to be defined carefully in terms of the unique successor of n and its successor and so on; see axioms N2, N3, N4.

[2] This sentence is how one would write it for most purposes! To be very careful we might instead say: If $f : S \to \{1\}$ then $x = f^{-1}(1)$ is the maximum, because if $y \in S$ then $y = x$. (The reason $y = x$ is because there is a unique natural number 1 and f is a function. That 1 is unique must be proven from N1,…,N5.)