Solutions to Midterm # 2

1. **(a)** *Definition*. For $N \in \mathbb{N}$, let $v_N = \sup\{s_n : n > N\}$. Then

$$\limsup s_n = \lim_{N\to\infty} v_N.$$

(b) *Proof*. Let $u_N = \inf\{s_n : n > N\}$. Because $S \subset T$ implies $\inf S \geq \inf T$, u_N is increasing. Since (s_n) is bounded, the set $\{s_n\}$ of all values from the sequence is also bounded. But then (u_N) is also a bounded sequence. Because there is a theorem that says that every bounded monotonic sequence converges, therefore (u_N) converges. By definition, $\lim_{N\to\infty} u_N = \liminf s_n$. Thus $\liminf s_n$ exists. □

2. *Proof*. Suppose (s_n) converges to $s \in \mathbb{R}$. Let $\epsilon > 0$. There exists N so that if $n > N$ then $|s_n - s| < \epsilon/2$. So now suppose $m, n > N$. We compute by the triangle inequality,

$$|s_m - s_n| = |s_m - s + s - s_n| \leq |s_m - s| + |s - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We have proven that (s_n) is Cauchy, by definition. □

3. *Definition*. We say f *is continuous at* x_0 if for all sequences (x_n) such that $x_n \in \mathrm{dom}(f)$ and such that $x_n \to x_0$, it follows that $f(x_n) \to f(x_0)$.

4. *Proof*. Consider the partial sums

$$s_N = \sum_{n=0}^{N} ar^n = a + ar + ar^2 + \cdots + ar^N.$$

Note that

$$rs_N - s_N = (ar + ar^2 + ar^3 + \cdots + ar^{N+1}) - (a + ar + ar^2 + \cdots + ar^N) = ar^{N+1} - a.$$

Thus $(r-1)s_N = a(r^{N+1} - 1)$ or

$$s_N = \frac{a(1 - r^{N+1})}{1 - r}.$$

Using the fact that $r^k \to 0$ as $k \to \infty$ if $|r| < 1$, we have

$$\sum_{n=0}^{\infty} ar^n = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \frac{a(1 - r^{N+1})}{1 - r} = \frac{a(1-0)}{1-r} = \frac{a}{1-r}.$$ □

5. **(a)** Let $S = 0.\overline{57}$. Then $100S = 57.\overline{57}$. By subtraction, $99S = 57$. Thus $S = 57/99$.

(b)

$$0.\overline{57} = \sum_{k=1}^{\infty} \frac{57}{100^k} = \sum_{n=0}^{\infty} \frac{57}{100}\left(\frac{1}{100}\right)^n.$$

In particular, $a = 57/100$ and $r = 1/100$.

(*Comment*. Thus $0.\overline{57} = a/(1-r) = 57/99$. This gives a different way to derive the result in part **(a)**.)

6. **(a)** The series converges. In fact,

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

is alternating, with terms $(-1)^n a_n$ where $a_n = 1/n$. Note $a_{n+1} \leq a_n$ for all n (i.e. a_n decreases) and also $a_n \to 0$. By the Alternating Series Test, the series converges.

(b) The series converges. First,

$$\frac{2}{n! + 7} \leq \frac{2}{n!}.$$

Next, the series

$$\sum_{n=0}^{\infty} \frac{2}{n!}$$

converges by the Ratio Test because

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2n!}{(n+1)!2} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

and $0 < 1$. Therefore, by the Comparison Test, the series

$$\sum_{n=0}^{\infty} \frac{2}{n! + 7}$$

also converges, because we know it is smaller than a convergent series.

7. *Proof.* Suppose the series $\sum a_n$ converges. That is, suppose that the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ converges. Let $s = \lim s_n$. Let $\epsilon > 0$. Choose N so that if $n > N$ then $|s_n - s| < \epsilon/2$. Then, noting $n+1, n+2 > N$ if $n > N$,

$$|s_{n+2} - s_{n+1}| = |s_{n+2} - s + s - s_{n+1}| \leq |s_{n+2} - s| + |s - s_{n+1}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

On the other hand,

$$a_{n+2} = \sum_{k=1}^{n+2} a_k - \sum_{k=1}^{n+1} a_k = s_{n+2} - s_{n+1}.$$

Thus we know that for any $\epsilon > 0$ there is N so that $n > N$ implies $|a_{n+2}| = |s_{n+2} - s_{n+1}| < \epsilon$. Thus $a_{n+2} \to 0$ so $a_n \to 0$. □

(*Comment*. In **7** you can also prove it by using the fact that the sequence of partial sums (s_n) is Cauchy, or, equivalently, by using the Cauchy criterion for series.)