Assignment #5

Due Monday 21 October, 2013 at the start of class

Please read sections 9, 10, 11, 12 of the textbook *Elementary Analysis*. Then do *all* of the following exercises. Turn them in on paper.

(The circled problem on your paper is the one you should also do in LaTeX and email to me at `elbueler@alaska.edu`.)

Exercise 9.11 (a).

Exercise 9.12.

Exercise 9.15.

Exercise 9.18 (a) and (b).

Exercise 10.2.

Exercise 10.5.

Exercise 10.6.

Exercise 10.9. (*Hint on* **(b)**: Use Theorem 10.2.)

Exercise 11.3. (On *this* problem you don't need to prove any of your claims.)

Exercise 11.7.

Exercise E3. In exercise 10.9 the sequence s_n is easily approximated using a calculator. I get these values:

$(1, 0.5, 1.667\times 10^{-1}, 2.083\times 10^{-2}, 3.472\times 10^{-4}, 1.005\times 10^{-7}, 8.652\times 10^{-15}, 6.550\times 10^{-29}, \dots)$

These numbers are going to zero very fast! This exercise illustrates that a standard approximation tool is effective because the *errors* it makes go to zero this fast.

(a) For a differentiable function f and a "first guess" x_1, Newton's method approximately solves $f(x) = 0$ by generating a sequence (x_n) from the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

The idea is that each successive x_n is a better approximation of the exact solution to $f(x) = 0$. Using the familiar derivatives facts on $f(x) = x^2 - 5$, use a calculator or computer to apply Newton's method to generate $x_2, \dots, x_7$ if $x_1 = 1$.

(b) Suppose $\hat{x}$ is the (exact) solution to $f(x) = 0$ closest to x_1. The absolute differences $e_n = |x_n - \hat{x}|$ are the *approximation errors* from Newton's method. Compute $e_1, \dots, e_6$ using your calculated results from **(a)**, and your knowledge of the value of $\hat{x}$ in **(a)**.[1]

(c) In most numerical analysis books[2] you will find a theorem like the following:

> *Theorem.* If f is twice-continuously-differentiable, if x_1 is sufficiently close to an exact solution $\hat{x}$ of $f(x) = 0$, and if $f(\hat{x}) \neq 0$, then Newton's method generates (x_n) that converges to $\hat{x}$. Furthermore, if $e_n = |x_n - \hat{x}|$ then
>
> $$\lim_{n\to\infty} \frac{e_{n+1}}{(e_n)^2} = C \qquad \text{where} \qquad C = \frac{|f''(\hat{x})|}{|f'(\hat{x})|}. \tag{2}$$

Suppose $f(x) = x^2 - 5$ and $x_1 = 1$ as in part **(a)**. What is C from equation (2)? As an approximate matter, are the calculated $e_1, \dots, e_6$ from part **(b)** behaving as claimed in this theorem?

(d) Suppose (s_n) is a nonnegative sequence and suppose $s_{n+1} \leq C s_n^2$ for some $C > 0$. Prove by induction that if $s_1 \leq 1/C$ then (s_n) is a decreasing sequence. Conclude that $s_n \leq 1/C$ for all n.

(e) Again suppose (s_n) is a nonnegative sequence and suppose $s_{n+1} \leq C s_n^2$ for some $C > 0$, but also assume that $s_1 < 1/C$ (strict inequality). Conclude using Theorem 10.2 and limit theorems from section 9 that $s_n \to 0$.

[1] *Caution.* If you claim $e_n = 0$ for some n then you are claiming $x_n = \hat{x}$. Avoid making such a claim unless it is true!

[2] For example, see page 85 of Greenbaum & Chartier, *Numerical Methods: Design, Analysis, and Computer Implementation of Algorithms*, Princeton University Press 2012.