Assignment #3

Due Monday 30 September, 2013 at the start of class

Please read Sections 7 and 8 of the textbook *Elementary Analysis*, but also review Sections 4, 5, and 6. Make sure you can do Exercise 7.3, but do not turn that in.[1] Then do the following exercises and turn them in on paper. The one circled problem on your paper is the one you should do in LaTeX and email to me.

Exercise E2. **(a)** I claimed in class that the set of real-coefficient rational functions

$$\mathbb{R}(x) = \left\{ \frac{p(x)}{q(x)} \,:\, p(x), q(x) \text{ are polynomials with real coefficients and } q(x) \neq 0 \right\}$$

is an ordered field. Of course rational functions should be familiar from precalculus, calculus, etc. State the formulas which define the standard way of adding and multiplying rational functions.

(b) You do not need to check this, but I assert that with the standard meanings you stated in part **(a)**, axioms A1,…,A4,M1,…,M4,DL apply. Now I define an order: We say

$$\frac{p(x)}{q(x)} \le \frac{r(x)}{s(x)}$$

if

$$\frac{a_0}{b_0} > 0 \quad \text{where} \quad \frac{r(x)}{s(x)} - \frac{p(x)}{q(x)} = \frac{a_0 x^n + \dots}{b_0 x^m + \dots} \quad \text{and} \quad a_0, b_0 \neq 0$$

with highest power n and m in numerator and denominator, respectively, or if the two rational functions are equal ($p(x)/q(x) = r(x)/s(x)$ for all x). That is, a rational function is less than or equal to another if the ratio of the leading-order coefficients from the numerator and denominator of the difference of the two rational functions, subtracted in the order given, is a nonnegative number. Show that this is a good definition by showing that if a rational function is written two different ways

$$\frac{u(x)}{v(x)} = \frac{a_0 x^n + \dots}{b_0 x^m + \dots} \quad \text{and} \quad \frac{u(x)}{v(x)} = \frac{\tilde{a}_0 x^{\tilde{n}} + \dots}{\tilde{b}_0 x^{\tilde{m}} + \dots},$$

with all of $a_0, b_0, \tilde{a}_0, \tilde{b}_0$ nonzero, then the ratios a_0/b_0 and $\tilde{a}_0/\tilde{b}_0$ of the coefficients of the highest powers are equal. You may use the fact that if a polynomial is zero then all of its coefficients are zero.

[1] Note that Exercise 7.3 asks you to recall calculations you did in Calculus II, wherein you probably did not *prove* the limits!

(c) Show that axiom O2 holds, that is,

$$\frac{p(x)}{q(x)} \leq \frac{r(x)}{s(x)} \quad \text{and} \quad \frac{p(x)}{q(x)} \geq \frac{r(x)}{s(x)} \quad \text{then} \quad \frac{p(x)}{q(x)} = \frac{r(x)}{s(x)}.$$

I claim that the other order axioms O1,O3,O4,O5 hold, but you don't have to show this.

(d) Show that x, thought of as a rational function (e.g. $x/1$), is greater than any constant polynomial (i.e. $c = c/1$). Thereby show that this ordered field is not Archimedean.

Exercise 4.8.

Exercise 4.11.

Exercise 4.16.

Exercise 5.4.

Exercise 5.6.

Exercise 7.4.

Exercise 8.1 (a) and (b). *Yes, do write formal proofs.*

Exercise 8.1 (c) and (d). *Yes, do write formal proofs.*

Exercise 8.3.