# Worksheet: Orthogonal functions and polynomials

**Definition.** Two vectors $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are *orthogonal* if

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = 0.$$

**Definition.** Two continuous functions $f(x)$ and $g(x)$ are *orthogonal* on the interval $[a, b]$ if

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx = 0.$$

(*The calculation* $\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$ *is called the* inner product *of the two functions.*)

**A.** Let $\mathbf{u} = (2, 1, 0)$ and $\mathbf{v} = (-\frac{1}{2}, 1, 1)$. Show $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Then sketch $\mathbf{u}, \mathbf{v}$ on three-dimensional axes, basing these vectors at the origin as usual, and indicate the right angle between $\mathbf{u}$ and $\mathbf{v}$.

**B.** Show $\mathbf{u} = (2, 3, -4, 1, 2)$ and $\mathbf{v} = (1, -1, 1, 3, 1)$ are orthogonal. (*I won't ask you to sketch because I don't know how. Neither do you.*)

**C.** Suppose $f(x) = x^k$ and $g(x) = x^l$ where $k \geq 0$ is an even integer and $l \geq 0$ is an odd integer. Show $f$ and $g$ are orthogonal on $[-1, 1]$.

**Definition.** Continuous functions $q_0(x), q_1(x), q_2(x), \ldots$ form an *orthogonal set* on $[a, b]$ if

$$\langle q_i, q_j \rangle = \int_a^b q_i(x) q_j(x)\, dx = 0 \quad \text{whenever } i \neq j.$$

**D.** Show the following polynomials form an orthogonal set on $[-1, 1]$:

$$q_0(x) = 1$$
$$q_1(x) = x$$
$$q_2(x) = 3x^2 - 1$$
$$q_3(x) = 5x^3 - 3x$$

(*The full list is* $\infty$*-ly long!*)

**E.** (*Extra Credit*) Show $\sin(mx)$ and $\sin(nx)$ are orthogonal on $[0, \pi]$ if $m \neq n$ are nonnegative integers. (*Hint: Use a trigonometric identity to do the integral.*) Sketch this.