Worksheet (*corrected*): Basic usage of ODE methods

An ordinary differential equation (ODE) initial value problem (IVP) is this problem:

$$y' = f(t, y), \quad y(t_0) = y_0$$

In a numerical method to solve this problem one first chooses a stepsize $h > 0$. The basic idea is that the derivative is replaced by a difference quotient: $y' \approx (y_{k+1} - y_k)/h$, but also the time advances by $t_{k+1} = t_k + h$. Then the method itself is a formula which determines a new value from an old value:

$$(*) \qquad y_{k+1} = \begin{pmatrix} \text{some formula in terms of } h, t_k, \text{ and } y_k, \\ \text{using } f \text{ and possibly its derivatives} \end{pmatrix}.$$

We call y_k the *current* value and y_{k+1} the *next* or *updated* value. You start the method knowing the value of y_0 so then you find y_1, then y_2, and so on.

A formula $(*)$ is called a *one step* method (section 11.2). There are also *multistep* methods using $y_k, y_{k-1}, y_{k-2}, \dots$ on the right side (section 11.3), but we will not cover them. A one step method can have multiple *stages* corresponding to the number of times f is evaluated in one step—see the "explicit midpoint method" below—but in a one step method each calculation starts fresh with only the value of y_k known.

A. For the problem

$$y' = t - y, \quad y(0) = 2$$

do two steps of *Euler's method* using $h = 1$:

$$y_{k+1} = y_k + h\, f(t_k, y_k)$$

B. The step sizes do not have to be equal. Redo the problem in part **A** using $h = 0.5$ on the first step and $h = 1.5$ on the second step.

C. For the same problem as in part **A**, do one $h = 2$ step of the *second-order Taylor method*:

$$y_{k+1} = y_k + h\,f(t_k, y_k) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\,f\right)(t_k, y_k).$$

D. Taylor methods have the disadvantage that you have to compute partial derivatives of $f(t, y)$, though that was easy in part **B**. The *(explicit) midpoint rule* is the two-stage formula

$$\begin{aligned} y_{k+1/2} &= y_k + \frac{h}{2} f(t_k, y_k) \\ y_{k+1} &= y_k + h\,f(t_{k+1/2}, y_{k+1/2}) \end{aligned}$$

For the same problem as in part **A**, do one $h = 2$ step.

E. Each of the above methods gives an estimate of $y(2)$ for the problem in part **A**, but in fact the exact solution is $y(t) = 3e^{-t} + t - 1$. (*Corrects error:* $2 \to 3$.) Compare accuracy of the methods.