Solutions to Worksheet on 3 methods to get the same polynomial

Vandermonde's construction. The linear system is

$$\begin{aligned} c_0 - c_1 + c_2 &= 3 \\ c_0 \qquad\qquad &= 7 \\ c_0 + 2c_1 + 4c_2 &= -2. \end{aligned}$$

Solve this to get $c_0 = 7$, $c_1 = 7/6$, and $c_2 = -17/6$. Thus

$$p(x) = 7 + \frac{7}{6}x - \frac{17}{6}x^2.$$

Lagrange's construction. First the Lagrange polynomials, which *only* use the x-coordinates of the points:

$$\begin{aligned} L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{1}{3}x(x-2), \\ L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -\frac{1}{2}(x+1)(x-2), \\ L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{1}{6}x(x+1). \end{aligned}$$

Thus

$$p(x) = x(x-2) - \frac{7}{2}(x+1)(x-2) - \frac{1}{3}x(x+1).$$

Newton's construction. The linear system is

$$\begin{aligned} a_0 \qquad\qquad &= 3 \\ a_0 + a_1 \qquad &= 7 \\ a_0 + 3a_1 + 6a_2 &= -2. \end{aligned}$$

Solve this to get $a_0 = 3$, $a_1 = 4$, and $a_2 = -17/6$. Thus

$$p(x) = 3 + 4(x+1) - \frac{17}{6}x(x+1).$$

Show the three polynomials are all the same. I show they are the same by converting all of them to "standard form" for polynomials. Thus I show that the Lagrange and Newton forms are each the same as the Vandermonde.

From Lagrange to standard:

$$p(x) = x^2 - 2x - \frac{7}{2}(x^2 - x - 2) - \frac{1}{3}x^2 - \frac{1}{3}x = 7 + \left(-2 + \frac{7}{2} - \frac{1}{3}\right)x + \left(1 - \frac{7}{2} - \frac{1}{3}\right)x^2 = 7 + \frac{7}{6}x - \frac{17}{6}x^2.$$

From Newton to standard:

$$p(x) = 3 + 4x + 4 - \frac{17}{6}(x^2 + x) = 7 + \frac{7}{6}x - \frac{17}{6}x^2.$$

Which is best? There is no simple answer. The standard form (Vandermonde) is the one we like for calculus, because we can use the power rule. The Lagrange form is mostly useful in theory. It is easy to understand the construction but then the algebra is nasty for most additional steps. The Newton form is the most efficient. Because of the triangular structure of the linear system it is easy to find the coefficients. From the coefficients it is easy to evaluate the polynomial, e.g. for plotting. Note that Horner's method can be applied to both the standard and Newton forms.