Solutions to Worksheet on 2 and 3 point Gaussian quadrature rules

Construct the $n = 2$ rule "naively". **(a)** The system simplifies to

$$\begin{aligned} w_1 + w_2 &= 2 \\ w_1 x_1 + w_2 x_2 &= 0 \\ w_1 (x_1)^2 + w_2 (x_2)^2 &= \frac{2}{3} \\ w_1 (x_1)^3 + w_2 (x_2)^3 &= 0 \end{aligned}$$

(b) With the substitution $x_1 = -a$, $x_2 = a$ the second equation is

$$-w_1 a + w_2 a = 0.$$

Since $a \neq 0$ so we can divide by it and get $w_1 = w_2$. The fourth equation says $-w_1 a^3 + w_2 a^3 = 0$, which is redundant now and can be ignored.

Then the first equation says

$$2w_1 = 2$$

so in fact $w_1 = w_2 = 1$. The third equation now says

$$2a^2 = \frac{2}{3}$$

from which it follows that $a = 1/\sqrt{3}$.

Thus the rule is

$$\int_{-1}^{1} f(x)\,dx \approx 1\,f\left(-\frac{1}{\sqrt{3}}\right) + 1\,f\left(\frac{1}{\sqrt{3}}\right).$$

Check degree of precision for the $n = 3$ point rule. The three integrals are exactly

$$\int_{-1}^{1} x^4\,dx = \frac{2}{5}, \qquad \int_{-1}^{1} x^5\,dx = 0, \qquad \int_{-1}^{1} x^6\,dx = \frac{2}{7}.$$

Applying the $n = 3$ rule for these integrals gives

$$\begin{aligned} \frac{5}{9}\left(-\sqrt{\frac{3}{5}}\right)^4 + \frac{8}{9}\,0^4 + \frac{5}{9}\left(\sqrt{\frac{3}{5}}\right)^4 &= 2\,\frac{5}{9}\left(\frac{3^2}{5^2}\right) = \frac{2}{5}, \\ \frac{5}{9}\left(-\sqrt{\frac{3}{5}}\right)^5 + \frac{8}{9}\,0^5 + \frac{5}{9}\left(\sqrt{\frac{3}{5}}\right)^5 &= -\frac{5}{9}\left(\sqrt{\frac{3}{5}}\right)^5 + \frac{5}{9}\left(\sqrt{\frac{3}{5}}\right)^5 = 0, \\ \frac{5}{9}\left(-\sqrt{\frac{3}{5}}\right)^6 + \frac{8}{9}\,0^6 + \frac{5}{9}\left(\sqrt{\frac{3}{5}}\right)^6 &= 2\,\frac{5}{9}\left(\frac{3^3}{5^3}\right) = \frac{6}{25}. \end{aligned}$$

Thus only the application to $f(x) = x^6$ is wrong, and $p = 5$ is indeed the degree of precision.

(*Note that* $2/7 = 0.28571$ *while* $6/25 = 0.24$, *so the result is sort of close anyway. But not exact.*)