Worksheet: 3 methods to get the same polynomial

Only two rules:
1. please *do not* use the internet or any book
2. please *do* talk to each other

Suppose we have three points:

$$(x_0, y_0) = (-1, 3), \qquad (x_1, y_1) = (0, 7), \qquad (x_2, y_2) = (2, -2).$$

There is a unique degree $n = 2$ polynomial $p(x)$ that goes through these points. This worksheet constructs this polynomial three different ways. However, it is the same polynomial so these apparently-different forms *must* be equivalent. The last step is to show that equivalence.

Vandermonde's construction. Assume

$$p(x) = c_0 + c_1 x + c_2 x^2.$$

Set up and solve a linear system of 3 equations in the 3 unknowns c_0, c_1, c_2. Solve this system and write down the resulting $p(x)$. As soon as you have a formula for $p(x)$, *stop*.

Lagrange's construction. Compute Lagrange's polynomials $L_0(x), L_1(x), L_2(x)$ for the x-coordinates x_0, x_1, x_2. (The is no reason to simplify the $L_i(x)$ here.) Then build $p(x)$. As soon as you have a formula for $p(x)$, *stop*.

Newton's construction. Assume

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1).$$

Set up and solve a linear system of 3 equations in the 3 unknowns a_0, a_1, a_2. As soon as you have a formula for $p(x)$, *stop*.

Show the three polynomials are all the same. You can do this by converting into whichever is your preferred form for $p(x)$.

Which is best? Primarily thinking in terms of higher-degree cases, give reasons why each of the three forms might be preferred over the others when using these polynomials for various purposes.