Worksheet: 2 and 3 point Gaussian quadrature rules

Only two rules:

1. please *do not* use the internet or any book
2. please *do* talk to each other

Consider a Gaussian quadrature rule on the standard interval $[-1, 1]$:

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i).$$

The idea is to choose the weights w_i and the nodes x_i to make the degree of precision as high as possible.

Construct the $n = 2$ rule "naively". (a) The $n = 2$ point rule is

$$\int_{-1}^{1} f(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2).$$

There are four unknowns w_1, w_2, x_1, x_2. We hope to make this rule exact for $f(x) = x^0, x^1, x^2, x^3$. Therefore, simplify the following four equations into a system which is as simple as possible. (*Especially: Compute right-hand sides!*)

$$w_1(x_1)^0 + w_2(x_2)^0 = \int_{-1}^{1} x^0\,dx$$

$$w_1(x_1)^1 + w_2(x_2)^1 = \int_{-1}^{1} x^1\,dx \qquad \overset{\text{simplifies to}}{\Longrightarrow}$$

$$w_1(x_1)^2 + w_2(x_2)^2 = \int_{-1}^{1} x^2\,dx$$

$$w_1(x_1)^3 + w_2(x_2)^3 = \int_{-1}^{1} x^3\,dx$$

(b) The above system is *nonlinear* so mere linear algebra does not solve it. However, the second and fourth equations above can be combined to show $x_1^2 = x_2^2$. (The rule is symmetric around zero.) Therefore let $x_1 = -a$ and $x_2 = a$ for $a > 0$. The second of the above equations then says $w_1 = w_2$; confirm this. Proceed to completely solve, and fill in the blanks at the bottom.

$$\int_{-1}^{1} f(x)\,dx \approx \_\_\_\, f(\_\_\_) + \_\_\_\, f(\_\_\_).$$

To construct higher-degree Gaussian quadrature rules one uses the roots of a Legendre polynomial as the nodes. In fact the weights and nodes are known for all n-point rules. The construction is in Lemma 5.3 and Theorem 5.4. This theory shows that the rule has degree of precision $p = 2n - 1$. The next part simply asks you to confirm this fact for the $n = 3$ rule.

Check degree of precision for the $n = 3$ point rule. The rule is

$$\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right).$$

Since this rule uses $n = 3$ points, the claim is that it has degree of precision $p = 2n - 1 = 5$. This means that it exactly integrates $f(x) = x^0, x^1, x^2, x^3, x^4, x^5$ but that it fails on $f(x) = x^6$.

To save you work, I have already checked it is exact for $f(x) = x^0, x^1, x^2, x^3$ (not shown). Confirm the three highest degrees, that is, apply the rule to $f(x) = x^4$, $f(x) = x^5$, and $f(x) = x^6$ and compare to the exact integrals.