Worksheet: Decimal and Binary Representations of Real Numbers

Only two rules:
1. *don't* use the internet
2. *do* talk to each other

You have used scientific notation many times, of course. It is a scheme for representing real numbers using decimal digits. Recalling how it works will help understand the binary scheme on the other side of this sheet.

System D-. Suppose "$\Diamond$" stands for either $+$ or $-$ and "$\square$" stands for a single decimal digit (i.e. from $\{0,1,2,3,4,5,6,7,8,9\}$). Consider the system of scientific notation below:

$$\Diamond\square.\square\square \times 10^{\Diamond\square}$$

(a) What are the largest and smallest numbers you can represent?

(b) What is the smallest *positive* number you can represent?

(c) How many *distinct* numbers can be represented? (*Be careful. This is so hard you may want to move on.*)

System D. Now suppose "$\hat{\square}$" stands for a single *positive* decimal digit (i.e. from $\{1,\dots,9\}$). Consider this closer-to-standard system of scientific notation:

$$\Diamond\hat{\square}.\square\square \times 10^{\Diamond\square}$$

(a) What are the largest and smallest numbers you can represent?

(b) What is the smallest *positive* number you can represent?

(c) How many *distinct* numbers can be represented? (*Easier.*)

(d) How can zero be represented?

System B16. On actual computers, real numbers are represented in binary, i.e. base 2. Suppose "$\Diamond$" stands for either $+$ or $-$, "$\hat{\square}$" stands for a single positive binary digit, and "$\square$" stands for any binary digit. Consider this scheme which is *close* to the IEEE 754 "half precision" standard. (*Normally we use IEEE double precision with 64 bits, but that's tedious here.*) Note there are 16 positions, each using 1 bit:

$$\Diamond\hat{\square}.\square\square\square\square\square\square\square\square\square\square_2 \times 2^{\Diamond\square\square\square\square_2}$$

(a) What are the largest and smallest numbers you can represent?

(b) What is the smallest *positive* number you can represent?

(c) How many *distinct* numbers can be represented?

(d) How can zero be represented?

(e) How do you use a bit ($\{0, 1\}$) to represent either $+$ or $-$, for use in the "$\Diamond$" locations?

(f) What is the gap between the smallest representable number larger than one and one itself? (*It is common to call this number "ϵ".*)

(g) Can you make the representation slightly more efficient by not "wasting" a bit which (almost?) never carries information?