Solutions to Midterm Exam

1. Here $f(x) = \sin x$, $n = 4$, and $x_0 = 0$. The particular value is $x = 0.5$ so by Taylor's theorem,

$$f(0.5) = p_4(0.5) + \frac{f^{(5)}(\xi)}{5!}(0.5 - 0)^5$$

where ξ is some number between $x_0 = 0$ and $x = 0.5$. But all we want is a bound for the error. We know $f^{(5)}(x) = \cos x$. Because $|\cos\theta| \le 1$ for any θ,

$$|f(0.5) - p_4(0.5)| = \left|\frac{\cos(\xi)}{120}(0.5)^5\right| = \frac{|\cos(\xi)|}{120 \cdot 2^5} \le \frac{1}{3840}.$$

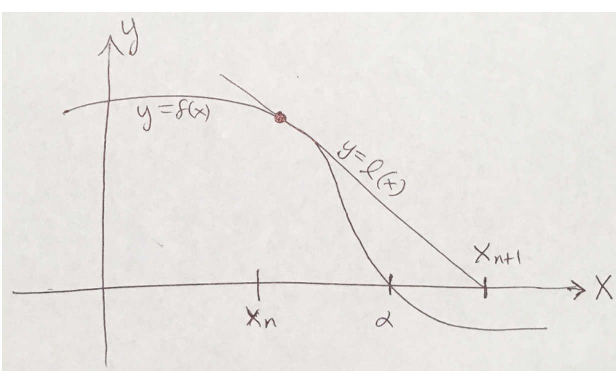
(*Full credit for a correct but unsimplified number.*) This upper bound is between 10^{-4} and 10^{-3}.

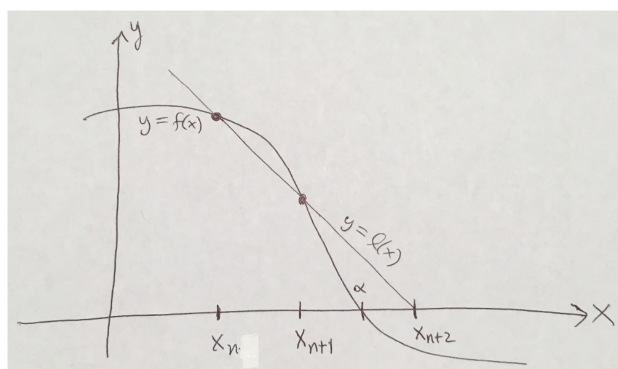
2. Theorem. Suppose $f \in C^{n+1}[a,b]$ and $x_0, x_1, \dots, x_n$ are distinct points in $[a,b]$. Then there is a unique polynomial $p_n(x)$ of degree at most n so that $p_n(x_i) = f(x_i)$ for $i = 0, 1, \dots, n$. Furthermore if $x \in [a,b]$ then there is $\xi \in [a,b]$ so that

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n).$$

(*You do not have to state the second sentence "Then there is ..." to receive full credit. I state it here to clarify the meaning of $p_n(x)$. What you* do *have to give correctly are the hypotheses and the form of the remainder term.*)

3. **(a)** sketch of Newton at left, and **(c)** sketch of secant at right





(b) As shown in the left sketch, we find the tangent line through the point $(x_n, f(x_n))$. It has slope $m = f'(x_n)$ so it is

$$\ell(x) = f(x_n) + f'(x_n)(x - x_n).$$

This line crosses the x-axis at x_{n+1}:

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n).$$

Solving for x_{n+1} then gives the Newton's method iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

4. Euler's method in general is $y_{n+1} = y_n + hf(t_n, y_n)$. In this case:

$$y_{n+1} = y_n + h(t_n - y_n^2).$$

The t-values are known. We have $t_0 = 2$, and spacing $h = 1$, so $t_1 = 3$ and $t_2 = 4$. Now we take two steps:

$$y_1 = y_0 + h(t_0 - y_0^2) = 1 + 1 \cdot (2 - 1^2) = 2,$$
$$y_2 = y_1 + h(t_1 - y_1^2) = 2 + 1 \cdot (3 - 2^2) = 1.$$

Thus the three points are $(t_0, y_0) = (2, 1)$, $(t_1, y_1) = (3, 2)$, and $(t_2, y_2) = (4, 1)$.

5. We apply Taylor's theorem with $n = 1$ and base point x:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(\xi)}{2!}h^2.$$

Here ξ is some value between x and $x + h$. Then divide by h and rearrange slightly:

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{f''(\xi)}{2}h.$$

To justify replacing the last term with "$+\mathcal{O}(h)$," subtract, take absolute values, and find an upper bound:

$$\left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| = \frac{|f''(\xi)|}{2}|h| \le C|h|.$$

Here the constant is $C = \frac{1}{2}\max_x |f''(x)|$. Thus

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \mathcal{O}(h).$$

6. **(a)** (*The best pseudocode would have more checks, e.g. whether $[a, b]$ is a bracket and whether $f(c)$ is exactly zero, but full credit is given for the following minimal version.*)

```
function z = bisection(f,a,b,n)

for k = 1 to n
    c = (a + b) / 2
    if f(a) * f(c) < 0
        b = c
    else
        a = c
    end
end
z = c
```

(b) Doing n steps of bisection reduces the length of the interval by a factor of 2^n, so we solve

$$|b_n - a_n| = \frac{1}{2^n}|b - a| = \frac{1}{2^n} \le 10^{-8}$$

for n. That is,

$$2^n \ge 10^8$$

or by taking the base 2 logarithm,

$$n \ge \log_2(10^8) = 8\log_2(10) = \frac{8\ln 10}{\ln 2}.$$

(*Any of these expressions gets full credit. With a calculator you'd find: $n \ge 26.575$.*)

(c) (*Here is a minimal but full-credit answer.*) Bisection is better than Newton because it does not need f'. Also, it is more robust because it maintains a bracket around the solution. Newton is better because it is faster, namely it converges quadratically once it gets close.

7. **(a)**

$$p(x) = 7 + x \cdot ((-2) + x \cdot (10 + x \cdot 4)).$$

(b)

```
function z = horner(c,x)

n = length(c) - 1
z = c(n+1)
for k = n downto 1                    %  "k = n:-1:1" in Matlab
    z = c(k) + x * z
end
```

8. Because $-x^2 < 0$ if $x \neq 0$, $e^{-x^2} < e^0 = 1$ if $x \neq 0$. So the maximum is at zero:

$$\|f\|_\infty = \max_{-1 \le x \le 1} |f(x)| = e^0 = 1.$$

(*You can also do calculus to find the maximum if you don't observe the above.*)

9. **(a)**

$$p_2(x) = 2\,\frac{(x-1)(x-3)}{(0-1)(0-3)} + (-1)\,\frac{(x-0)(x-3)}{(1-0)(1-3)} + 7\,\frac{(x-0)(x-1)}{(3-0)(3-1)}.$$

(b) The Newton form is

$$p_2(x) = a_0 + a_1(x-0) + a_2(x-0)(x-1).$$

Seeking $p_2(x_i) = y_i$ gives three equations in three unknowns:

$$\begin{aligned} a_0 &= 2, \\ a_0 + a_1(1-0) &= -1, \\ a_0 + a_1(3-0) + a_2(3-0)(3-1) &= 7. \end{aligned}$$

Thus $a_0 = 2$, then $a_1 = (-1-2)/1 = -3$, and $a_2 = (7 - 2 - 3(-3))/6 = 14/6 = 7/3$. Thus

$$p_2(x) = 2 + (-3)(x-0) + \frac{7}{3}(x-0)(x-1).$$

10. (*Was mislabeled as "***9***." Also, it should have said "*$x_{n+1} = \frac{1}{3}e^{2\,x_n}$*" with an "n" subscript.*)

Let $g(x) = \frac{1}{3}e^{-2x}$. Note that g is decreasing and $g(0) = \frac{1}{3}$ and $g(2) = \frac{1}{3}e^{-4} < \frac{1}{3}$ are both in $[0,2]$ so $g([0,2]) \subset [0,2]$. Also,

$$g'(x) = -\frac{2}{3}e^{-2x}$$

so if $x \ge 0$ then $|g'(x)| = \frac{2}{3}e^0 = \frac{2}{3} < 1$. Let $\gamma = \frac{2}{3}$. By the Mean Value Theorem we have shown

$$|g(x) - g(y)| \le \gamma |x - y|$$

for all x, y in $[0,2]$. From the Fixed-point Convergence Theorem it follows that $x_{n+1} = g(x_n)$ converges for all x_0 in $[0,2]$.

(*It converges to the unique* α *in* $[0,2]$ *such that* $\alpha = g(\alpha)$*. But you don't need to say this.*)

(*One thing that is helpful, and worth a point of credit, is a sketch.*)

Extra Credit. Note that $\lim_{k\to\infty} 3^{-k} = 0$ so $\lim_{n\to\infty} x_n = 0$ also. Let $\alpha = 0$. Then

$$\frac{x_{n+1} - \alpha}{(x_n - \alpha)^2} = \frac{3^{-(2^{n+1})}}{\left(3^{-(2^n)}\right)^2} = \frac{3^{-(2\,2^n)}}{3^{-2\,(2^n)}} = 1.$$

Since $1 \in (0, \infty)$, by definition x_n converges quadratically to $\alpha = 0$.