Assignment #8

Due Friday 30 November, 2012 at the start of class

In your solutions to all problems, remember to:

- show me your calculations,
- show me the MATLAB code or commands, if used, and
- check that you have answered the question.

These problems are on polynomial and piecewise-polynomial interpolation. Refer to Chapter 8 of the textbook.

Problem I. **(a)** Find n so that degree n polynomial interpolation of $f(x) = \cos(3x)$, using equally-spaced points on $[0,2]$, gives a maximum approximation error $|f(x) - p(x)|$ which is less than 10^{-6} on $[0,2]$. (*Hints*: Use Theorem 8.4.1. You may use the in-class conservative upper bound method, "option 2.")

(b) Compute n again for $f(x)$ replaced by $g(x) = \cos(25x)$.

(c) Use MATLAB's `polyfit` and `polyval` to find the actual smallest n needed to approximate $f(x) = \cos(3x)$ to within 10^{-6}. That is, for various n smaller than your result in part **(a)**, actually evaluate the maximum of $|f(x) - p(x)|$.

Problem II. Let $f(x) = 7 - 2(x-2)^3 + x^2$. Consider 5 equally-spaced points on the interval $[-3,1]$, and suppose $p(x)$ is the unique degree 4 (or lower) interpolating polynomial for $f(x)$ using these points. Apply Theorem 8.4.1 to give an upper bound on the maximum of the approximation error $|f(x) - p(x)|$ on the interval. Now explain why the value of $|f(x) - p(x)|$ is actually obvious, and the upper bound you computed is not surprising!

Problem III. **(a)** At the bottom of page 198 is an inequality that describes the error from the piecewise-linear interpolant $\ell(x)$ for $f(x)$ on $[a,b]$. Suppose we have equally-spaced points $a = x_0 < x_1 < \cdots < x_n = b$ with spacing $h = x_i - x_{i-1}$. Then:

$$|f(x) - \ell(x)| \overset{*}{\leq} \frac{Mh^2}{8} \qquad \text{for all } x \in [a,b].$$

In this inequality we are assuming $f''(x)$ exists and is bounded by the number M, so that $|f''(x)| \leq M$ for all $x \in [a,b]$. Use inequality $*$ to find n so that $|f(x) - \ell(x)| \leq 10^{-6}$ for $x \in [0,2]$ if $f(x) = \cos(3x)$.

(b) Suppose you need to compute approximate values of $f(x) = \cos(3x)$ at x in-between the interpolation points. In several sentences, explain which technique requires more computation in practice, using $p(x)$ from problem **I (c)** or $\ell(x)$ in part **(a)** of this problem. Assume that $p(x)$ and $\ell(x)$ have been pre-computed and stored. Do not worry about memory usage but do consider floating point operations.

Problem IV. Derive composite Simpson's rule as follows: Suppose $f(x)$ is continuous on $[a, b]$ and $a = x_0 < x_1 < \cdots < x_n = b$ are $n + 1$ equally-spaced points with spacing $h = (b - a)/n$. Assume n is even so there are an odd number of points. Using Lagrange's form of the interpolating polynomial, give a formula for $q(x)$, the piecewise-quadratic polynomial interpolant on $[a, b]$. In particular, give the formula for $q(x)$ on $[x_{i-2}, x_i]$ for $i = 2, 4, 6, \ldots, n$, and note that $q(x)$ is a continuous function built from $n/2$ quadratic polynomials defined on subintervals $[x_0, x_2]$, $[x_2, x_4]$, ... Now integrate like I did in class for the trapezoid rule:

$$\int_a^b f(x)\,dx \approx \int_a^b q(x)\,dx = \sum_{k=1}^{n/2} \int_{x_{2k-2}}^{x_{2k}} q(x)\,dx = \ldots$$

Simplify your formula! Specifically, pay attention to when h, or multiples of h, appear in your formulas. The result should be

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\right].$$

This problem is from Chapter 8 of the textbook.

Problem Exercise 16 on page 211.

The remaining problems are from Chapter 10 of the textbook.

Problem Exercise 1 on page 248.

Problem Exercise 6 on page 249.