How to put a polynomial through points

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MATH 310 Numerical Analysis

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purpose

These notes are an online replacement for the 14 September class of Math 310 (Fall 2012), while Bueler is away.

*The topics here are also covered in Chapter 8 of the text (Greenbaum & Chartier). The emphasis here is on **how** to put a polynomial through points. When we get to Chapter 8 we will address the "how good" question.*

an example of the problem

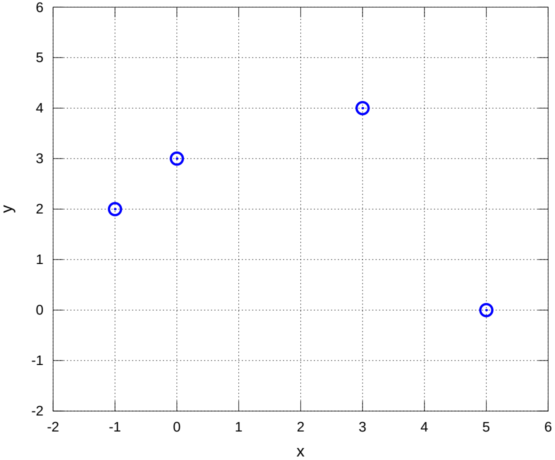
- suppose you have a function $y = f(x)$ which goes through these points:

$$(-1, 2), \quad (0, 3), \quad (3, 4), \quad (5, 0)$$

- the x-coordinates of these points are not equally-spaced!
  - in these notes I will *never* assume the x-coordinates are equally-spaced
- let us name these points (x_i, y_i), for $i = 1, 2, 3, 4$
- there is a polynomial $P(x)$ of degree 3 which goes through these points
- we will build it concretely
- we will show later that there is only one such polynomial

a picture of the problem

- figure below shows the points
- as stated, we suppose that they are values of a function $f(x)$
- but we don't *see* that function



how to find $P(x)$

- $P(x)$ is the degree 3 polynomial through the 4 points
- a standard way to write it is:

$$P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

- *note*: there are 4 unknown coefficients and 4 points
  - degree $n-1$ polynomials have the right length for n points
- the facts "$P(x) = y$" for the given points gives 4 equations:

$$\begin{aligned}
c_0 + c_1(-1) + c_2(-1)^2 + c_3(-1)^3 &= 2 \\
c_0 + c_1(0) + c_2(0)^2 + c_3(0)^3 &= 3 \\
c_0 + c_1(3) + c_2(3)^2 + c_3(3)^3 &= 4 \\
c_0 + c_1(5) + c_2(5)^2 + c_3(5)^3 &= 0
\end{aligned}$$

- MAKE SURE that you are clear on how I got these equations, and that you can do the same thing in an example with different points or different polynomial degree

a linear system

- you can solve the equations by hand ... that would be tedious
- we want to automate the process
- we have a great matrix-vector tool, namely MATLAB
- I recognize the system has a matrix form "$A\mathbf{v} = \mathbf{b}$":

$$\begin{bmatrix} 1 & -1 & (-1)^2 & (-1)^3 \\ 1 & 0 & 0^2 & 0^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 5 & 5^2 & 5^3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 0 \end{bmatrix}$$

- (a known square matrix A) $\times$ (an unknown vector $\mathbf{v}$) = (a known vector $\mathbf{b}$)
- I am not simplifying the numbers in the matrix ... because:
    - a machine can do that, and
    - the pattern in the matrix entries is clear if they are unsimplified
- MAKE SURE you can convert from the original "fit a polynomial through these points" question into the matrix form "$A\mathbf{v} = \mathbf{b}$"

how to *easily* find $P(x)$

- MATLAB is designed to solve linear systems ... easily!
- enter the matrix and the known vector into MATLAB:

```
>> A = [1 -1 (-1)^2 (-1)^3; 1 0 0^2 0^3; 1 3 3^2 3^3; 1 5 5^2 5^3]
A =
     1    -1     1    -1
     1     0     0     0
     1     3     9    27
     1     5    25   125
>> b = [2; 3; 4; 0]
b =
   2
   3
   4
   0
```

- solve the linear system to get $\mathbf{v} = [c_0\ c_1\ c_2\ c_3]$:

```
>> v = A \ b
v =
   3.000000
   0.983333
  -0.066667
  -0.050000
```

- so the polynomial is $P(x) = 3 + 0.983333x - 0.066667x^2 - 0.05x^3$

notes on matrices and vectors in MATLAB

- you enter matrices like A by rows
  - spaces separate entries
  - semicolons separate rows
- column vectors like $\mathbf{b}$ are just matrices with one column
  - to quickly enter column vectors use the transpose operation:

```
>> b = [2 3 4 0]'
b =
     2
     3
     4
     0
```

- to solve the system $A\mathbf{v} = \mathbf{b}$ we "divide by" the matrix: $\mathbf{v} = A^{-1}\mathbf{b}$
- …but this is *left* division, so MATLAB makes it into a single-character operation, the *backslash* operation:

```
>> v = A \ b
```

- the forward slash does not work because of the sizes of the matrix and the vector are not right:

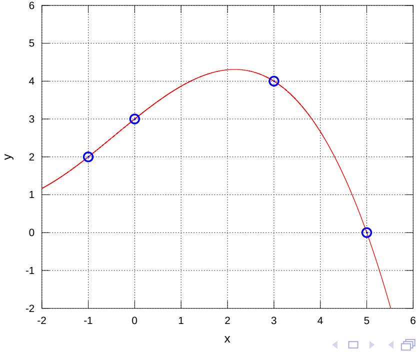
```
>> v = b / A   % NOT CORRECT for our A and b; wrong sizes
```

did we solve the problem?

- the polynomial we found had better go through the points:

```
>> 3.000000 + 0.983333*(-1) - 0.066667*(-1)^2 -0.050000*(-1)^3
ans =  2
>> 3.000000 + 0.983333*(0) - 0.066667*(0)^2 -0.050000*(0)^3
ans =  3
>> 3.000000 + 0.983333*(3) - 0.066667*(3)^2 -0.050000*(3)^3
ans =  4.0000
>> 3.000000 + 0.983333*(5) - 0.066667*(5)^2 -0.050000*(5)^3
ans = -1.0000e-05
```

- a graph is convincing, too:



- suppose we have n points (x_i, y_i) with distinct x-coordinates
  - for example, if $n = 4$ we have points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)
- then the polynomial has degree one less: the polynomial $P(x)$ which goes through the n points has degree $n - 1$
- the polynomial has this form:

$$P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$$

- the equations which determine $P(x)$ say that *the polynomial goes through the points*:

$$P(x_i) = y_i \qquad \text{for} \quad i = 1, 2, \dots, n$$

- written out there are n equations of this form:

$$c_0 + c_1 x_i + c_2 x_i^2 + \cdots + c_{n-1} x_i^{n-1} = y_i \qquad \text{for} \quad i = 1, 2, \dots, n$$

- the n coefficients c_i are unknown, while the x_i and y_i are known

the pattern in the matrix, for the general case

- as a matrix:

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ & \vdots & & \ddots & \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}$$

- and $\mathbf{b}$ is a column vector with entries y_i: $\mathbf{b} = [y_1 \quad y_2 \quad \dots y_n]'$
- as before, this gives a system of n equations, $A\mathbf{v} = \mathbf{b}$
- the matrix A is called a *Vandermonde matrix*, from about 1772

Vandermonde matrix, built-in

- actually, Vandermonde matrices are already built-in to MATLAB
- for example, the Vandermonde matrix A for our original four points $(-1,2),(0,3),(3,4),(5,0)$ is

```
>> vander([-1 0 3 5])
ans =
    -1      1     -1      1
     0      0      0      1
    27      9      3      1
   125     25      5      1
```

- two comments:
  - oops! the columns are in reversed order, compared to our choice
  - note that *only* the x-coordinates are needed to build A, and not the y-coordinates
- we easily fix the column order to agree with our earlier ordering using "`fliplr`", which stands for "flip left-to-right":

```
>> A = fliplr(vander([-1 0 3 5]))
A =
     1     -1      1     -1
     1      0      0      0
     1      3      9     27
     1      5     25    125
```

Vandermonde matrix method for polynomial interpolation

- thus a complete code to solve our 4 point problem earlier is:

```
A = fliplr(vander([-1 0 3 5]));
b = [2 3 4 0]';
v = A \ b
```

- after the coefficents `v` are computed, they form $P(x)$ this way:

$$P(x) = \texttt{v(1)} + \texttt{v(2)}\, x + \texttt{v(3)}\, x^2 + \cdots + \texttt{v(n)}\, x^{n-1}$$

- thus we can plot the 4 points and the polynomial this way:

```
plot([-1 0 3 5],[2 3 4 0],'o','markersize',12)
x = -2:0.01:6;  P = v(1) + v(2)*x + v(3)*x.^2 + v(4)*x.^3;
hold on, plot(x,P,'r'), hold off
xlabel x, ylabel y
```

- this was the "convincing" graph, shown earlier

on the cost of solving Vandermonde matrix problems

- now, often we want to do a polynomial fit problem like this *many times for different data*
- so, is it quick? here are some facts to know about solving these systems:
  - if there are n points then the matrix A has n rows and n columns
  - *internally in* MATLAB, the linear system $A\mathbf{v} = \mathbf{b}$ is solved by Gaussian elimination
  - Gaussian elimination does about $\frac{2}{3}n^3$ arithmetic operations (i.e. additions, subtractions, multiplications, divisions) to solve such a linear system
- so finding the coefficients of the polynomial $P(x)$ through n points takes about n^3 operations
- but then you need more operations to *evaluate* that polynomial, which is what you usually do with it

“new” idea: Newton’s form

- before Vandermonde there was already a good, practical idea
  - an old idea of Newton, perhaps about 1690
- the idea is to write the polynomial through the data $P(x)$ *not* using the “monomials” $1, x, x^2, x^3, \dots, x^{n-1}$,
- ...but instead to use a form of the polynomial which includes the x-coordinates of the data points:

$$\begin{aligned} P(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)(x - x_2) + \\ \dots + c_{n-1}(x - x_1)(x - x_2)\dots(x - x_{n-1}) \end{aligned}$$

- do you see why this helps?

Newton's form example: 4 points

- with the $n = 4$ points $(-1, 2), (0, 3), (3, 4), (5, 0)$ we can write

$$P(x) = c_0 + c_1(x+1) + c_2(x+1)(x) + c_3(x+1)(x)(x-3)$$

- this polynomial must go through the four points, so:

$$\begin{aligned}
&c_0 && = 2 \\
&c_0 + c_1(0+1) && = 3 \\
&c_0 + c_1(3+1) + c_2(3+1)(3) && = 4 \\
&c_0 + c_1(5+1) + c_2(5+1)(5) + c_3(5+1)(5)(5-3) && = 0
\end{aligned}$$

- note that lots of terms are zero!
- the system of equations has the form

$$M\mathbf{w} = \mathbf{b}$$

where M is a triangular matrix, $\mathbf{b}$ is the same as in the Vandermonde form, and $\mathbf{w}$ has the unknown coefficients:

$$\mathbf{w} = [c_0\ c_1\ c_2\ c_3]'$$

Newton's form example, cont.

- can you solve this by hand?
- *yes*: find c_0 from first equation, then c_1 from second equation, etc.
- I get $c_0 = 2$, $c_1 = 1$, $c_2 = -1/6$, $c_3 = -1/20$, so

$$P(x) = 2 + (x+1) - \frac{1}{6}(x+1)(x) - \frac{1}{20}(x+1)(x)(x-3)$$

- MAKE SURE you can do this yourself, on a similar example

Newton's form example, cont.[2]

- so we have a concrete polynomial, but not in standard form:

$$P(x) = 2 + (x+1) - \frac{1}{6}(x+1)(x) - \frac{1}{20}(x+1)(x)(x-3)$$

- an uninteresting calculation puts it in standard form:

$$\begin{aligned} P(x) &= 3 + \frac{59}{60}x - \frac{1}{15}x^2 - \frac{1}{20}x^3 \\ &= 3 + 0.983333x - 0.066667x^2 - 0.05x^3 \end{aligned}$$

- which is exactly the same polynomial we found earlier

Newton's form for polynomial interpolation: example code

- the advantage of the Newton form is that a *triangular* matrix M is created
  - which makes it easier to solve the system by hand
  - only $O(n^2)$ operations are needed to solve the system
  - the polynomial comes out in a non-standard form but it is just as easy to evaluate at a point
- for now here is a short code to solve the 4 point problem:

newt4.m

```
% NEWT4  Compute P(x) using the Newton form, for 4 points.

n = 4;  x = [-1 0 3 5]';  y = [2 3 4 0]';  % the points

M = zeros(n,n);  % makes M the right size
% form M by columns
M(:,1) = ones(n,1);
for j=2:n
  M(j:n,j) = M(j:n,j-1) .* (x(j:n) - x(j-1));
end
b = y;

w = M \ b  % w has the coefficients of the polynomial:
% P(x) = w1 + w2 (x-x1) + w3 (x-x1) (x-x2) + w4 (x-x1) (x-x2) (x-x3)
```

Newton's form shows there is a unique interpolating polynomial

- for both Vandermonde and Newton matrix approximations we build an invertible matrix, so in each case there is exactly one solution
- this is easiest to see from the general Newton form matrix:

$$M = \begin{bmatrix} 1 & & & & \\ 1 & (x_2 - x_1) & & & \\ 1 & (x_3 - x_1) & (x_3 - x_1)(x_3 - x_2) & & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & (x_n - x_1) & (x_n - x_1)(x_n - x_2) & \dots & (x_n - x_1)(x_n - x_2)\dots(x_n - x_{n-1}) \end{bmatrix}$$

- the diagonal entries are all nonzero as long as the x-coordinates are distinct
- we can calculate the determinant of this Newton form matrix M
- because the matrix is triangular, the determinant is the product of the diagonal:

$$\det M = \prod_{i>j} (x_i - x_j) \neq 0$$

- so the polynomial $P(x)$ always exists and is unique

Lagrange's idea: no systems at all!

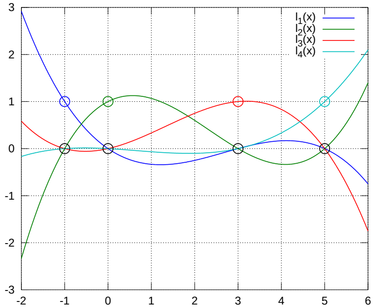
- another new idea
- given the same $n = 4$ points $(-1, 2), (0, 3), (3, 4), (5, 0)$
- Lagrange and others, by about 1800, knew how to write down four polynomials, now called the *Lagrange polynomials*, corresponding to the x-coordinates $x_1, \dots, x_4$:

$$
\begin{aligned}
\ell_1(x) &= \frac{(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} = \frac{x(x-3)(x-5)}{(-1)(-4)(-6)} \\
\ell_2(x) &= \frac{(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} = \frac{(x+1)(x-3)(x-5)}{(1)(-3)(-5)} \\
\ell_3(x) &= \frac{(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} = \frac{(x+1)(x)(x-5)}{(4)(3)(-2)} \\
\ell_4(x) &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} = \frac{(x+1)(x)(x-3)}{(6)(5)(2)}
\end{aligned}
$$

- the *pattern* needs attention: **I.** the numerator and denominator have the same pattern, but the denominator is a constant with no variable "x"; **II.** $\ell_i(x)$ has no "$(x - x_i)$" factor in the numerator, nor "$(x_i - x_i)$" factor in the denominator; **III.** as long as the x_i are distinct, we never divide by zero

Lagrange's idea: polynomials which "hit one point"

- why is this helpful?
- consider a plot of $\ell_1(x)$, $\ell_2(x)$, $\ell_3(x)$, $\ell_4(x)$:



- a crucial pattern emerges:

  the polynomial $\ell_i(x)$ has value 0 at all of the x-values of the points, except that it is 1 at x_i

- MAKE SURE make sure you can find the Lagrange polynomials if I give you the x-values of n points

Lagrange's idea, cont.

- the picture on the last page illustrates what is generally true of the Lagrange polynomials:

$$\ell_i(x_j) = \begin{cases} 1, & j = i, \\ 0, & \text{otherwise.} \end{cases}$$

- also, the Lagrange polynomials for the 4 points are each of degree 3
- so why does this help find $P(x)$?
- recall that we have values y_i which we want the polynomial $P(x)$ to "hit"
- that is, we *want* this to be true for each i:

$$P(x_i) = y_i$$

- *thus the answer is*:

$$P(x) = y_1\ell_1(x) + y_2\ell_2(x) + y_3\ell_3(x) + y_4\ell_4(x)$$

Lagrange's idea, cont.[2]

- *wait*, why is this the answer?:

$$P(x) \overset{*}{=} y_1\ell_1(x) + y_2\ell_2(x) + y_3\ell_3(x) + y_4\ell_4(x)$$

- because $P(x)$ *is* of degree three, as a linear combination of degree 3 polynomials, and
- because:

$$\begin{aligned} P(x_1) &= y_1\ell_1(x_1) + y_2\ell_2(x_1) + y_3\ell_3(x_1) + y_4\ell_4(x_1) \\ &= y_1 \cdot 1 + y_2 \cdot 0 + y_3 \cdot 0 + y_4 \cdot 0 \\ &= y_1, \end{aligned}$$

and

$$\begin{aligned} P(x_2) &= y_1\ell_1(x_2) + y_2\ell_2(x_2) + y_3\ell_3(x_2) + y_4\ell_4(x_2) \\ &= y_1 \cdot 0 + y_2 \cdot 1 + y_3 \cdot 0 + y_4 \cdot 0 \\ &= y_2, \end{aligned}$$

and so on

Lagrange's idea, cont.[3]

- on the last slide we saw that $P(x_i) = y_i$ because the polynomials $\ell_i(x)$ help "pick out" the point x_i in the general expression $*$ on the last slide
- we can say this more clearly using summation notation:
  - the polynomial is a sum of the Lagrange polynomials with coefficients y_i:

$$P(x) = \sum_{i=1}^{4} y_i \ell_i(x)$$

  - when we plug in one of the x-coordinates of the points, we get only one "surviving" term in the sum:

$$P(x_j) = \sum_{i=1}^{4} y_i \ell_i(x_j) = y_j \cdot 1 + \sum_{i \neq j} y_i \cdot 0 = y_j$$

returning to our 4-point example

- for our 4 concrete points $(-1,2),(0,3),(3,4),(5,0)$, we can slightly-simplify the Lagrange polynomials we have computed already:

$$\ell_1(x) = -\frac{1}{24}x(x-3)(x-5)$$
$$\ell_2(x) = +\frac{1}{15}(x+1)(x-3)(x-5)$$
$$\ell_3(x) = -\frac{1}{24}(x+1)(x)(x-5)$$
$$\ell_4(x) = +\frac{1}{60}(x+1)(x)(x-3)$$

- so the polynomial which goes through our points is

$$\begin{aligned} P(x) = &-(2)\frac{1}{24}x(x-3)(x-5) + (3)\frac{1}{15}(x+1)(x-3)(x-5) \\ &- (4)\frac{1}{24}(x+1)(x)(x-5) + (0)\frac{1}{60}(x+1)(x)(x-3) \end{aligned}$$

- a tedious calculation simplifies this to

$$P(x) = 3 + \frac{59}{60}x - \frac{1}{15}x^2 - \frac{1}{20}x^3,$$

which is exactly what we found earlier

so, is the Lagrange scheme a good idea?

- for n points $\{(x_i, y_i)\}$ we have the following nice formulas which "completely answer" the polynomial interpolation problem:

$$\ell_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

$$P(x) = \sum_{i=1}^{n} y_i \ell_i(x)$$

- note "$\prod$" is a symbol for a product, just like "$\sum$" is a symbol for sum
- we solve no linear systems and we just write down the answer! *yeah*!
- is this scheme a good idea in practice?
  NOT REALLY!

so, is the Lagrange scheme a good idea? cont.

- we have seen that actually using the formulas to find a familiar form for $P(x)$ is ... awkward
- the problem with the Lagrange form is that even when we write down the correct linear combination of Lagrange polynomials $\ell_i(x)$ to give $P(x)$, we do not have quick ways of getting:
  - either the coefficients a_i in the standard form,
  $$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$
  - or the values of the polynomial $P(x)$ at locations $\bar{x}$ in between the x_i:
  $$P(\bar{x}) = \bar{y}$$
- generally-speaking, the output values of a polynomial are the desired numbers; this is the purpose of polynomial *interpolation*
- **moral**: sometimes a *formula* for the answer is less useful than an algorithm that leads to the numbers you actually want
- ... and we'll get back to that!

conclusion: how to do polynomial interpolation

- the problem is to find the degree $n-1$ polynomial $P(x)$ which goes through n given points (x_i, y_i)
- we have three methods, all of which do the job:
  - the Vandermonde matrix method,
  - the Newton polynomial form, and its triangular matrix method,
  - and Lagrange's direct formula for the polynomial
- the first two require solving linear systems, while the last does not
  - Lagrange's direct formula requires us to simplify like crazy
  - Newton gives easier linear systems (triangular) than does Vandermonde
  - MATLAB makes solving linear systems easy anyway
- later in the course:
  - we will address how accurate polynomial interpolation is
  - we will get to one more algorithm, Neville's algorithm, which gets the polynomial values but skips finding any coefficients