Solutions to Midterm Exam #1

1. Find the solution to the initial value problem:

$$\frac{dy}{dx} + 2y = e^{-x}, \qquad y(-1) = e.$$

Solution. This ODE is linear, and $\mu = e^{2x}$, so we need to solve:

$$\frac{d}{dx}\left(e^{2x}\,y\right) = e^{2x}e^{-x} = e^x,$$
$$e^{2x}\,y = \int e^x\,dx = e^x + C,$$
$$y(x) = e^{-2x}\left(e^x + C\right) = e^{-x} + Ce^{-2x}.$$

But $e = y(-1) = e^{+1} + Ce^{+2} = e + Ce^2$ so $0 = Ce^2$ so $C = 0$. The solution to the ODE IVP is $y(x) = e^{-x}$, which is easy to check.

2. Use Euler's method to approximate the solution to the initial value problem at the points $x = 0.1$ and $x = 0.2$, with steps of size $h = 0.1$:

$$\frac{dy}{dx} = x + y, \qquad y(0) = 1.$$

Solution. Here $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$. Euler's method says $y_{n+1} = y_n + hf(x_n, y_n) = y_n + h(x_n + y_n)$. Note $y_0 = 1$. Thus

$$y_1 = y_0 + h(x_0 + y_0) = 1 + 0.1(0 + 1) = 1.1 \approx y(0.1),$$
$$y_2 = y_1 + h(x_1 + y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.1 + 0.12 = 1.22 \approx y(0.2).$$

3. A brine solution of salt flows at a constant rate of 8 L/min into a large tank that initially held 100 L of brine solution in which was dissolved 0.5 kg of salt. The solution inside the tank is kept well stirred and flows out of the tank at the same rate. If the concentration of salt in the brine entering the tank is 0.05 kg/L, determine the mass of salt in the tank after t minutes, *and* set up an equation to determine when the concentration of salt in the tank reaches 0.02 kg/L.

This problem is exactly 3.2 #1. The first equation below is the main concern in getting the problem right. Both sides of that equation have units of kg/min.

Solution. Let $x(t)$ be the *mass* of salt in the tank, measured in kg, at time t, measured in minutes. Thus $x(0) = 0.5$. The container model is to compute the rate of change as the difference of input and output:

$$\frac{dx}{dt} = 8\,(0.05) - 8\left(\frac{x}{100}\right),$$

which simplifies to $dx/dt = 0.4 - 0.08x$. This is separable and it is linear. Treated as separable,

$$\int \frac{dx}{0.4 - 0.08x} = \int dt,$$
$$\frac{1}{-0.08}\ln|0.4 - 0.08x| = t + C,$$
$$0.4 - 0.08x = \tilde{A}e^{-0.08t},$$
$$x(t) = \frac{0.4}{0.08} + Ae^{-0.08t} = 5 + Ae^{0.08t}.$$

But we know $x(0) = 0.5$ so $A = -4.5$ and so $\boxed{x(t) = 5 - 4.5e^{0.08t}}$. This answers the first part of the question.

To "set up an equation ...", note we want the *concentration* of salt to reach a certain level. That is, we want $x(t)/100$ to be equal to 0.02. We solve this for t:

$$\frac{5 - 4.5e^{0.08t}}{100} = 0.02.$$

Of course you can simplify this equation, but that is not essential or asked ...

4. (a). Show that the following equation is exact:

$$\frac{t}{y}\,dy + (1 + \ln y)\,dt = 0.$$

I will think of t as independent and y as dependent, though that is not essential.

Solution. Let $M(t, y) = 1 + \ln y$, $N(t, y) = t/y$. Then

$$\frac{1}{y} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} = \frac{1}{y}.$$

Therefore there is a function $F(t, y)$ so that along a level curve $F(t, y) = C$ we have $(t/y)\,dy + (1 + \ln y)\,dt = 0$. (In part **(b)** we seek F.)

(b). Solve the ordinary differential equation in part **(a)**. Write the solution as an explicit formula for $y(t)$.

Solution. From "$\partial F/\partial t = M$" we have

$$F(t, y) = \int M\,dt + g(y) = \int 1 + \ln y\,dt + g(y) = t + (\ln y)t + g(y).$$

On the other hand, from "$\partial F/\partial y = N$" we have

$$0 + \frac{1}{y}t + g'(y) = \frac{t}{y}.$$

This says $g'(y) = 0$, so, because we need only one such g at this stage, choose $g(y) = 0$. Then $F(t, y) = t + (\ln y)t$. The implicit solution of the ODE is as a level curve of F: $t + (\ln y)t = C$. This can be solved for y:

$$y(t) = e^{(C-t)/t} = e^{(C/t)-1}.$$

5. (a). Show that $6y^{1/3} - x^2 = 7$ defines an implicit solution to the ODE

$$\frac{dy}{dx} = xy^{2/3}$$

Solution. Implicit differentiation of "$6y^{1/3} - x^2 = 7$":

$$6\left(\frac{1}{3}\right)y^{-2/3}\frac{dy}{dx} - 2x = 0,$$

$$2y^{-2/3}\frac{dy}{dx} = 2x,$$

$$\frac{dy}{dx} = xy^{2/3}.$$

(b). The textbook states this theorem about the existence of solutions to ordinary differential equation initial value problems:

Theorem. Given the initial value problem $dy/dx = f(x, y)$, $y(x_0) = y_0$, assume that f and $\partial f/\partial y$ are continuous functions in a rectangle $R = \{a < x < b,\ c < y < d\}$ that contains the point (x_0, y_0). Then the initial value problem has a unique solution $y = \phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.

For the ODE in part **(a)**, are there any points in the plane where this theorem does not apply, so that at such points there might be either no solution or more than one solution? Describe all such problematic points, or state that there are none.

Solution. First, $f(x, y) = xy^{2/3}$ in **(a)**, which is continuous everywhere in the (x, y) plane; *think about this as it is not automatic.* But $\partial f/\partial y = (2/3)xy^{-1/3}$, so $\partial f/\partial y$ is not defined, and thus not continuous, where $y = 0$. It follows that all points $(x, y) = (x, 0)$ are "problematic" in the above sense.

We only know that the theorem does not apply *at these points. But I think it is also the case that at these points there is more than one solution passing through the point. Can you check that?*

6. (a). The ODE $dy/dx = xy$ has the direction field shown below. Sketch the solution to this ODE which passes through $(x_0, y_0) = (0, 2)$.

Solution. The solution through $(0, 2)$ is plotted in figure 1. So is the solution through $(1, 1)$ described by part **(b)**.

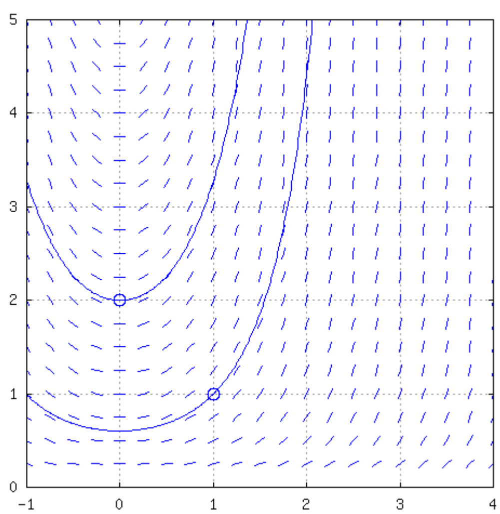

FIGURE 1. Solution curves for problem **6**, parts **(a)** and **(b)**.

(b). Solve the following initial value problem—*give an exact formula $y(x)$ for the solution!*—and then sketch the solution you find on the direction field above:

$$\frac{dy}{dx} = xy, \qquad y(1) = 1.$$

Solution. The equation is separable (and linear):

$$\int \frac{dy}{y} = \int x\,dx,$$

$$\ln|y| = \frac{1}{2}x^2 + C,$$

$$y(x) = Ae^{x^2/2}.$$

The initial condition says $1 = Ae^{1/2}$, so

$$y(x) = e^{(x^2-1)/2}.$$

Now, it would be relatively hard to sketch this just from the formula, but the direction field in part **(a)** is exactly what this solution follows. And it goes through $(1, 1)$; it is shown in figure 1.

7. Find the general solution

$$\frac{dz}{dt} = \frac{t-5}{z^2-z}$$

Solution. Separable:

$$\int (z^2 - z)\, dz = \int (t-5)\, dt,$$

$$\frac{1}{3}z^3 - \frac{1}{2}z^2 = \frac{1}{2}t^2 - 5t + C.$$

Since we are allowed to write the answer implicitly, we are done.

Extra Credit. Let $y(x)$ be the solution to this initial value problem:

$$\frac{dy}{dx} + 2xy = 1, \qquad y(0) = 0.$$

Write an integral for $y(1)$. Use this integral, and a Taylor series approximation, to estimate $y(1)$ within 1%.

Solution. This equation is first-order linear with $\mu(x) = e^{x^2}$. So we write

$$\frac{d}{dx}\left(e^{x^2} y(x)\right) = e^{x^2}.$$

We can include the initial condition $y(0) = 0$, and our ending x-value of $x = 1$, as we integrate:

$$\int_0^1 \frac{d}{dx}\left(e^{x^2} y(x)\right) dx = e^{1/2}y(1) - e^0 y(0) = \int_0^1 e^{x^2}\, dx,$$

$$y(1) = e^{-1/2}\int_0^1 e^{x^2}\, dx.$$

This is the first part of the question.

Now we approximate this integral using the Taylor (Maclaurin) series for e^x. First we plug x^2 into the variable for the series for e^t,

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots,$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \dots,$$

and then we integrate the result:

$$\int_0^1 e^{x^2}\, dx = \int_0^1 \left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \dots\right) dx$$

$$= 1 + \frac{1}{3} + \frac{1}{2\cdot 5} + \frac{1}{6\cdot 7} + \frac{1}{24\cdot 9} + \dots$$

$$\approx 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} + \frac{1}{216}.$$

This last value is credibly within 1% of the correct answer because of the rapid decrease of the terms in the series. In particular, the next term is $5! \cdot 11 = 120 \cdot 11 = 1320$. Thus the approximate value for $y(1)$ is

$$y(1) = \frac{1}{\sqrt{e}}\left(1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} + \frac{1}{216}\right).$$