Selected Solutions to Assignment #9

These problems were graded at 3 points each for a total of 27 points.

7.2 #4. An integration-by-parts:

$$\mathscr{L}\left\{te^{3t}\right\}(s) = \int_0^\infty e^{-st}te^{3t}\,dt = \int_0^\infty te^{(3-s)t}\,dt = t(3-s)^{-1}e^{(3-s)t}\Big|_{t=0}^{t=\infty} - \int_0^\infty (3-s)^{-1}e^{(3-s)t}\,dt$$
$$= -(3-s)^{-1}\int_0^\infty e^{(3-s)t}\,dt = -(3-s)^{-2}e^{(3-s)t}\Big|_{t=0}^{t=\infty} = +(3-s)^{-2} = \frac{1}{(s-3)^2}.$$

This function of s is defined for $s > 3$; we used "$s > 3$" in evaluating the limit at $t = \infty$. The result agrees with a rule in table 7.1.

7.2 #14. From the table,

$$\mathscr{L}\left\{5 - e^{2t} + 6t^2\right\}(s) = \frac{5}{s} - \frac{1}{s-2} + \frac{12}{s^3}.$$

7.2 #22. The function is piecewise continuous on $[0, 10]$. The graph is in Figure 1.

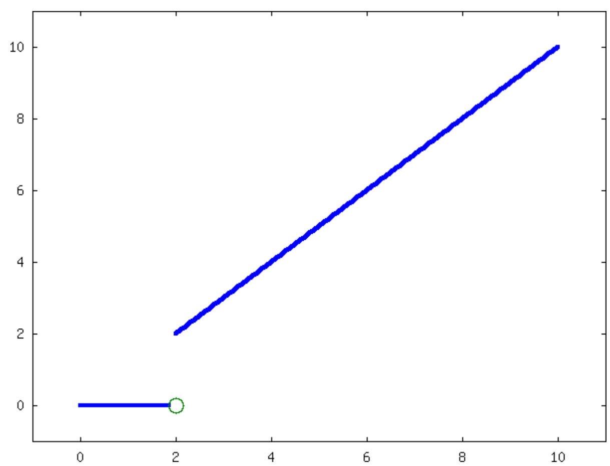

FIGURE 1. Sketch of $f(t)$ from #22 in section 7.2.

7.3 #10. Here we use the last rule in table 7.2 to turn the problem into one covered by the last entry in table 7.1, which means we use the quotient rule:

$$\mathscr{L}\left\{te^{2t}\cos 5t\right\}(s) = -\frac{d}{ds}\left(\mathscr{L}\left\{e^{2t}\cos 5t\right\}(s)\right) = -\frac{d}{ds}\left(\frac{s-2}{(s-2)^2+5^2}\right) = \frac{(s-2)^2-5^2}{((s-2)^2+5^2)^2}.$$

7.3 #24. **(a)** The translation property in question says $\mathscr{L}\{f\}(s-a) = \mathscr{L}\left\{e^{at}f\right\}(s)$. So we write, also using the third line of table 7.1,

$$\mathscr{L}\left\{e^{at}t^n\right\}(s) = \mathscr{L}\{t^n\}(s-a) = \frac{n!}{\sigma^{n+1}}\Big|_{\sigma=s-a} = \frac{n!}{(s-a)^{n+1}}.$$

My point in writing "σ" is merely to indicate that we first transform from t to a new variable and only then replace the transform variable with "$s - a$".

(b) Using formula (6), and noting that the powers of $(s-a)$ are easy to differentiate any number of times,

$$\begin{aligned}\mathscr{L}\left\{e^{at}t^n\right\}(s) &= (-1)^n \frac{d^n}{ds^n}\left(\mathscr{L}\left\{e^{at}\right\}(s)\right) = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s-a}\right) \\ &= (-1)^n(-1)(-2)\cdots(-(n-1))(-n)\frac{1}{(s-a)^{n+1}} = (-1)^n(-1)^n\frac{n!}{(s-a)^{n+1}} = \frac{n!}{(s-a)^{n+1}}.\end{aligned}$$

7.3 #30. Let $Y(s) = \mathscr{L}\{y\}(s)$ and $G(s) = \mathscr{L}\{G\}(s)$. To find the transfer function $H(x) = Y(s)/G(s)$ we assume initial conditions $y(0) = y'(0) = 0$ and apply the rules for Laplace transforms of derivatives to the differential equation:

$$s^2Y(s) - 0 - 0 + 5(sY(s) - 0) + 6Y(s) = G(s),$$

or $(s^2+5s+6)Y(s) = G(s)$. Therefore

$$H(s) = \frac{Y(s)}{G(s)} = \frac{1}{s^2+5s+6}.$$

7.4 #2. From table 7.1 with $b = 2$,

$$\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \mathscr{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \sin 2t.$$

7.4 #14. Factor the denominator completely. The problem is to find A, B, C in this equation:

$$\frac{-8s^2-5s+9}{(s-2)(s-1)(s+1)} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{s+1}.$$

Clear denominators. Substituting $s = 2$ gives $A = -11$. Substituting $s = 1$ gives $B = 2$. Substituting $s = -1$ gives $C = 1$. Thus:

$$\frac{-8s^2-5s+9}{(s+1)(s^2-3s+2)} = -\frac{11}{s-2} + \frac{2}{s-1} + \frac{1}{s+1}.$$

7.4 #24: A good example, *but* not assigned. Note $s^2 - 4s + 13$ does not factor over real numbers, but we can complete the square: $s^2 - 4s + 13 = (s-2)^2 + 3^2$. The easiest form is similar to example 7,

$$\frac{7s^2-41s+84}{\left((s-2)^2+3^2\right)(s-1)} = \frac{A(s-2)+3B}{(s-2)^2+3^2} + \frac{C}{s-1}.$$

Clear denominators. Substitute $s = 1$ gives $C = 5$. Substituting $s = 2$ gives $30 = 3B + 9C$ so $B = -5$. Substituting $s = 0$, for example, gives $84 = 2A - 3B + 13C$ so $A = 2$. Thus

$$\frac{7s^2-41s+84}{\left((s-2)^2+3^2\right)(s-1)} = 2\frac{(s-2)}{(s-2)^2+3^2} - 5\frac{3}{(s-2)^2+3^2} + 5\frac{1}{s-1}.$$

The inverse Laplace transform is

$$f(t) = 2e^{2t}\cos 3t - 5e^{2t}\sin 3t + 5e^t.$$

7.4 #26. The partial fraction form is

$$\frac{7s^3-2s^2-3s+6}{s^3(s-2)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s-2}.$$

Clearing denominators,

$$7s^3 - 2s^2 - 3s + 6 = A(s-2) + Bs(s-2) + Cs^2(s-2) + Ds^3.$$

Substitute $s - 0$ to find $A = -3$. Substitute $s = 2$ to find $D = 6$. Substitute $s = 1$ and $s = -1$, for example, to get this system of equations

$$\begin{aligned}B + C &= 1 \\ B - C &= -1\end{aligned}$$

Get $B = 0$ and $C = 1$. Thus

$$\mathscr{L}^{-1}\left\{\frac{7s^3-2s^2-3s+6}{s^3(s-2)}\right\} = \mathscr{L}^{-1}\left\{-\frac{3}{2}\frac{2}{s^3} + 0 + \frac{1}{s} + 6\frac{1}{s-2}\right\} = 1 - \frac{3}{2}t^2 + 6e^{2t}.$$