Selected Solutions to Assignment #8

These problems were graded at 3 points each for a total of 24 points.

5.2 #2. Rewrite as

$$Dx - 3y \overset{(i)}{=} 0$$
$$-2x + (D+1)y \overset{(ii)}{=} 0.$$

Combine by $2(i) + D(ii)$ to get: $D(D+1)y - 6y = 0$ or $y'' + y' - 6y = 0$. This second order equation has auxiliary equation $r^2 + r - 6 = 0$ or $(r+3)(r-2) = 0$ so $r = -3, +2$. Thus

$$x(t) = -c_1 e^{-3t} + (3/2)c_2 e^{2t}$$
$$y(t) = c_1 e^{-3t} + c_2 e^{2t}.$$

Note that once $y(t)$ is found we find $x(t)$ most easily by (ii): $x = (1/2)(y' + y) = (1/2)(-3c_1 e^{-3t} + 2c_2 e^{2t} + c_1 e^{-3t} + c_2 e^{2t})$. By setting $C_1 = -c_1$ and $C_2 = (3/2)c_2$, our solution is equivalent to

$$x(t) = C_1 e^{-3t} + C_2 e^{2t}$$
$$y(t) = -C_1 e^{-3t} + (2/3)C_2 e^{2t}.$$

5.2 #8. The approach is similar to above. Combine $(D+4)(i) - (D-1)(ii)$ to get an equation for x: $(D+4)(D-3)[x] - (D-1)(D+1)[x] = (D+4)t - (D-1)1$ or $(D - 12 + 1)[x] = 1 + 4t + 1$ or

$$x' - 11x = 4t + 2.$$

This first order linear equation has solution by integrating factors. Namely,

$$\frac{d}{dt}\left(e^{-11t}x(t)\right) = (4t+2)e^{-11t}$$
$$e^{-11t}x(t) = \int (4t+2)e^{-11t}\,dt$$
$$e^{-11t}x(t) = -(4t+2)\frac{e^{-11t}}{11} + \int 4\frac{e^{-11t}}{11}\,dt = -\frac{1}{11}(4t+2)e^{-11t} - \frac{4}{11}\frac{e^{-11t}}{11} + C$$

So

$$x(t) = -\frac{1}{11}(4t+2) - \frac{4}{121} + C\,e^{11t} = -\frac{4}{11}t - \frac{26}{121} + Ce^{11t}.$$

How to get $y(t)$? Either of the original equations is a first-order differential equation for $y(t)$. Choosing the first equation, and simplifying I get:

$$y' - y = -\frac{1}{11}t - \frac{34}{121} - 8Ce^{11t}.$$

This is first order linear. The solution is

$$y(t) = \frac{1}{11}t + \frac{45}{121} - \frac{8}{10}Ce^{11t} + c_2 e^{t}.$$

Note that the "C" in the formulas for $x(t)$ and $y(t)$ is the same number. The equations are coupled, so their constants are related.

5.2 #20. Write as

$$(D-2)x - y = -e^{2t}$$
$$-x + (D-2)y = 0$$

Combine these to eliminate x: $(D-2)(D-2)x - x = 0$ or $x'' - 4x' + 3x = 0$. This has auxiliary equation $r^2 - 4r + 3 = (r-3)(r-1) = 0$ so $x(t) = c_1 e^{3t} + c_2 e^t$. The first equation then gives y: $y(t) = c_1 e^{3t} - c_2 e^t + e^{2t}$.

The initial conditions determine c_1, c_2:

$$1 = x(0) = c_1 + c_2$$
$$-1 = y(0) = c_1 - c_2 + 1,$$

so $c_1 = -1/2$ and $c_2 = 3/2$. It follows that

$$x(t) = -\frac{1}{2}e^{3t} + \frac{3}{2}e^t$$
$$y(t) = -\frac{1}{2}e^{3t} - \frac{3}{2}e^t + e^{2t}.$$

5.3 #5. See back of text.

5.3 #10. First we write the single second-order ODE as a system of first-order ODEs:

$$y' = v$$
$$v' = -tv - y.$$

Here is a method for using MATLAB or OCTAVE as a calculator to quickly find the Euler's method answer. Note that the up-arrow gets the last command so it is easy to repeatedly enter the same thing, even if that thing is mildly complicated:

```
>> h=0.25;
>> t=0, y=1, v=0
  t = 0
  y = 1
  v = 0
>> ynew=y+h*(v); vnew=v+h*(-t*v-y); t=t+h, y=ynew, v=vnew
  t =  0.25000
  y =  1
  v = -0.25000
>> ynew=y+h*(v); vnew=v+h*(-t*v-y); t=t+h, y=ynew, v=vnew
  t =  0.50000
  y =  0.93750
  v = -0.48438
>> [AGAIN]
>> ynew=y+h*(v); vnew=v+h*(-t*v-y); t=t+h, y=ynew, v=vnew
  t =  1
  y =  0.65186
  v = -0.73889
```

Thus, in particular, $y(1) \approx 0.65186$ by Euler's method with $h = 0.25$.

(*The exact answer is hard to come by. Using* $h = 0.1$ *gives* $y(1) \approx 0.61852$ *and using* $h = 0.05$ *gives* $y(1) \approx 0.61160$*. Probably the value of* $y(1)$ *is about* 0.61)

5.3 #16. On this one I used the online applet at `http://www.csun.edu/~hcmth018/SysEu.html`. I entered "`2*x-y`" in the first box, "`3*x-6*y`" in the second box and then $x_0 = 0$, $y_0 = -2$, $b = 1$, $n = 8$. Note that $1/8 = 0.125 = h$. The result is:

```
t_n            x_n            y_n
    0.00000        0.00000       -2.00000
    0.12500        0.25000       -3.50000
    0.25000        0.75000       -6.03125
    0.37500        1.69141      -10.27344
    0.50000        3.39844      -17.34424
    0.62500        6.41608      -29.07800
    0.75000       11.65485      -48.48048
    0.87500       20.62862      -80.47027
    1.00000       35.84455     -133.08723
```

How accurate is this anyway? In this case we can compare to the exact values of $x(t)$ and $y(t)$ because the actual solutions, a pair of functions of time, are given. The easiest way to compare is again to use MATLAB or OCTAVE to evaluate the formulas $x(t)$ and $y(t)$ at the same list of t values. Thus, from the exact solutions, the above numbers *should* be:

```
>> t=0:0.125:1; x=exp(5*t)-exp(3*t); y=exp(3*t)-3*exp(5*t);
>> [t' x' y']
ans =
    0.00000     0.00000    -2.00000
    0.12500     0.41325    -4.14975
    0.25000     1.37334    -8.35403
    0.37500     3.44060   -16.48224
    0.50000     7.70080   -32.06579
    0.62500    16.23908   -61.75887
    0.75000    33.03335  -118.07551
    0.87500    65.63527  -224.51494
    1.00000   128.32762  -425.15394
```

The errors, the differences between the approximation and the exact values, grow rapidly in time.

Euler's method does better with a smaller h value, however. (A method like Runge-Kutta can do much better even with the same h.) Here is Euler again but with h smaller by a factor of four. I have plotted the approximate and exact answers in Figure 1. Euler does better but the error is still obvious.

```
>> h = 1/32;
>> t = 0:h:1;  xexact = exp(5*t)-exp(3*t);  yexact = exp(3*t)-3*exp(5*t);
>> x(1) = 0;  y(1) = -2;
>> for n=1:32,  x(n+1) = x(n)+h*(2*x(n)-y(n));  y(n+1) = y(n)+h*(3*x(n)+6*y(n));  end
>> plot(t,xexact,t,yexact,t,x,'o',t,y,'o')
```

5.4 #5. See back of text.

5.4 #10. The critical points are the solutions of

$$0 = x^2 - 1$$
$$0 = xy.$$

Thus the critical points are $(-1, 0)$ and $(+1, 0)$. The xy-phase plane equation is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{xy}{x^2 - 1}.$$

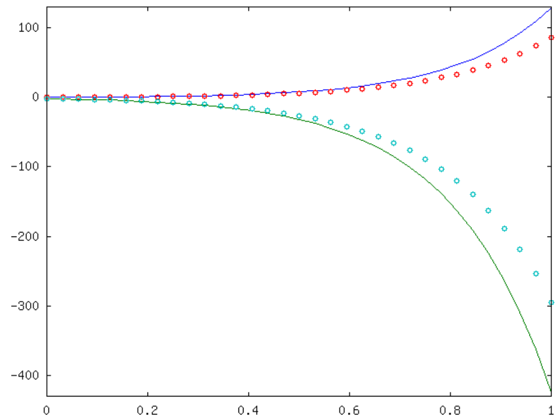

FIGURE 1. Exact (curve) and approximate (dots) solution to #16 in section 5.3.

This equation is separable:

$$\frac{dy}{y} = \frac{x}{x^2-1}\,dx$$

$$\ln|y| = \int \frac{x}{x^2-1}\,dx = \frac{1}{2}\ln\left|x^2-1\right| + C$$

$$y(x) = \begin{cases} A_0\sqrt{1-x^2}, & -1 \le x \le 1, \\ A_1\sqrt{x^2-1}, & |x| \ge 1. \end{cases}$$

The number A_0 and A_1 do not have to be the same. These solutions meet the critical points.

There are two solutions (i.e. trajectories in the phase plane) that are semicircles. Namely if $-1 \le x \le 1$ and $A_0 = 1$ we have the upper semicircle of the unit circle. For the same x values and $A_0 = -1$ we have the lower semicircle of the unit circle. Figure 2 shows these solutions plus the direction field for the differential equation $\frac{dy}{dx} = \frac{xy}{x^2-1}$, in the xy-plane.

It is pretty easy to show that the other solutions are pieces of ellipses and of hyperbolas.

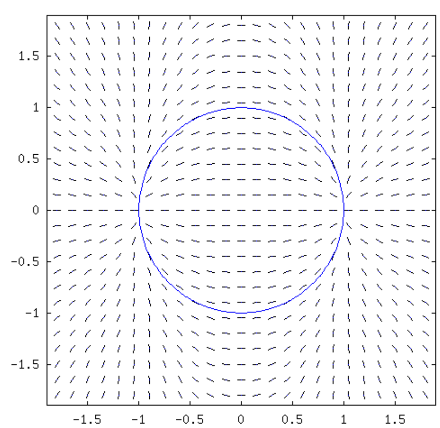

FIGURE 2. Semicircle solutions in xy-phase plane, to #10 in section 5.4.