# Selected Solutions to Assignment #7

*These problems were graded at 3 points each, except for 4.9 #4 which was 6 points, for a total of 21 points.*

**4.7 #33.** The answer "$Cte^{-t}$" is in the back of the book. To get this, put the equation in standard form so that you can identify $p(t) = (t-1)/t = 1 - t^{-1}$. Thus

$$W = Ce^{-\int_{t_0}^{t}(1-\tau^{-1})\,d\tau} = Ce^{-t+\ln t + C_1} = C_2 e^{-t} e^{\ln t} = C_2 t e^{-t}.$$

**4.8 #2.** Here $m = 1$, $b = 0$, and $k = -6y$. Note that the sign of $k$ depends on the sign of $y$. When $y < 0$ this is a realistic spring with positive spring constant.

But when $y > 0$ it is not a realistic spring. In this case the "spring" doesn't oppose you when you displace it from equilibrium. Thus if the initial condition has $y > 0$ then the spring produces positive feedback, sending $y(t)$ to infinity in finite time. There isn't even damping to slow it down.

**4.8 #6.** First put the equation in standard form for the energy integral lemma, $y'' = -(k/m)y$, so $f(y) = -(k/m)y$. Finding the antiderivative ($F' = f$),

$$F(y) = -\frac{k}{m}\left(\frac{1}{2}y^2\right) = -\frac{k}{2m}y^2.$$

The energy integral lemma says

$$E(t) = \frac{1}{2}(y')^2 - F(y) = \frac{1}{2}(y')^2 + \frac{k}{2m}y^2 = K$$

for some constant $K$. Thus, multiplying through by $2m$ and identifing "$2mK$" as the constant,

$$m(y')^2 + ky^2 = \text{constant}$$

**4.8 #11.** Here $m = 1$ and $k = 1$ so that part is boring. But

$$b = -\left[1 - (y')^2\right] = (y')^2 - 1$$

So this means that $b$ is negative when $y'$ is small in magnitude but then $b$ switches to positive when $y'$ gets big. In figures 4.24 and 4.25 the initial value for $y'$ is not to big. What happens is that the "damping" is instead "pulling" at the beginning, so the oscillation increases in magnitude. But then eventually the $y'$ value is big enough in magnitude, especially around the times when $y = 0$, so that the damping term is really acting like damping, so the oscillation stops accelerating. These several forces eventually balance out so that the cycles keep having the same magnitude.

Note that a steady oscillation is achieved even though there is no energy conservation and also no driving (external) force.

**4.9 #4.** Here "equation of motion" means solution to the initial value problem. The auxiliary equation is $r^2 + br + 64 = 0$, which gives roots

$$r_1, r_2 = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - 64}.$$

The rest of the problem is in cases, of course.
$b = 0$: *Underdamped: no damping!* $r_1, r_2 = \pm i8$ so $y(t) = c_1 \cos 8t + c_2 \sin 8t$. From initial values, $y(t) = \cos 8t$.
$b = 10$: *Underdamped.* $r_1, r_2 = -5 \pm i\sqrt{39}$ so $y = e^{-5t}\left(c_1 \cos\sqrt{39}t + c_2 \sin\sqrt{39}t\right)$. From initial values, $y(t) = e^{-5t}\left(\cos\sqrt{39}t + \frac{5}{\sqrt{39}}\sin\sqrt{39}t\right)$.

$b = 16$: *Critically damped.* $r_1, r_2 = r = -8$ so $y = (c_1 + c_2 t)\,e^{-8t}$. From initial values, $y(t) = (1 + 8t)\,e^{-8t}$.
$b = 20$: *Overdamped.* $r_1, r_2 = -4, -16$ so $y = c_1 e^{-4t} + c_2 e^{-16t}$. From initial values, $y(t) = \frac{4}{3}e^{-4t} - \frac{1}{3}e^{-16t}$.

Instead of a sketch, figure 1 shows the plot from a short MATLAB/OCTAVE program. The program itself is at `http://www.dms.uaf.edu/~bueler/exer4pt9_4.m`, and is printed below.

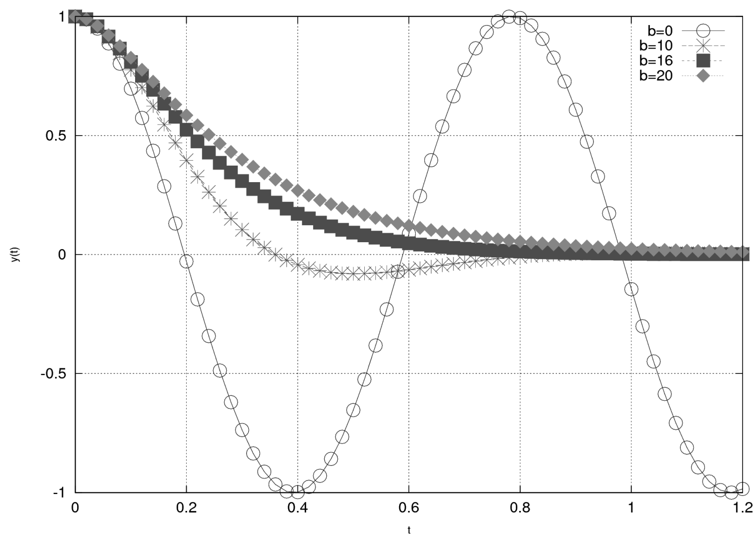


FIGURE 1. Sketch of $b = 0, 10, 16, 20$ cases, for #4 in section 4.9.

```
%% produce "sketch" for exercise #4 in section 4.9
t = 0:.02:1.2;
y0 = cos(8*t);
y10 = exp(-5*t) .* ( cos(sqrt(39)*t) + (5/sqrt(39))*sin(sqrt(39)*t) );
y16 = exp(-8*t) .* ( ones(size(t)) + 8 * t );
y20 = (4/3) * exp(-4*t) - (1/3) * exp(-16*t);
plot(t,y0,'o-','markersize',12,t,y10,'*-','markersize',12,...
     t,y16,'s-','markersize',12,t,y20,'d-','markersize',12)
legend("b=0","b=10","b=16","b=20")
xlabel t, ylabel('y(t)'), grid on
```

**4.9 #8**. Here we measure in meters, seconds, kilograms, and Newtons. There is no external force. Thus $m = 20$, $k = 200$, and $b = 140$, and the equation is

$$20y'' + 140y' + 200y = 0$$

which simplifies to $y'' + 7y' + 10y = 0$. The auxiliary equation is $r^2 + 7r + 10 = 0$, with roots $r_1, r_2 = -2, -5$, so the general solution is $y(t) = c_1 e^{-2t} + c_2 e^{-5t}$.

The initial conditions (meters and seconds!) are $y(0) = 0.25$ and $y'(0) = -1$. Thus the solution is

$$y(t) = \frac{1}{12}e^{-2t} + \frac{1}{6}e^{-5t}.$$

*This function is always positive!* Thus the mass never actually returns to its equilibrium position $y = 0$, though it decays exponentially to that value.