MATH 302 Differential Equations (Bueler) 24 March 2009

Selected Solutions to Assignment #6

These problems were graded at 3 points each for a total of 21 points.
(*The Group Project on A#6 is treated as a separate 10 point assignment.*)

4.4 #10. The auxiliary equation $r^2 + 2r - 1 = 0$ has roots $r = (-2 \pm \sqrt{4+4})/2 = -1 \pm \sqrt{2}$. On the other hand, the right hand side (nonhomogeneity) is $10 = 10t^0 e^{0t}$, so $s = 0$ in form (14) because $r = 0$ is not a root of the auxiliary equation. (*Note you can just check it is not:* $0^2 + 2(0) - 1 \neq 0$.) Thus we use

$$y_p(t) = t^0 (A_0) e^{0t} = A_0.$$

Substituting into the differential equation gives

$$(A_0)'' + 2(A_0)' - A_0 = 10$$

which is "$-A_0 = 10$". Thus $y_p(t) = -10$.

4.4 #12. The equation is first order but the same undermined coefficients approach works. (*You can also find a particular solution by treating this equation as first order linear and using the techniques of section 2.3.*) The auxiliary equation is $2r + 1 = 0$. The right hand side is of the form $3t^2 = Ct^m e^{rt}$ with $m = 2$ and $r = 0$. Since $r = 0$ is not a root (solution) of $2r + 1 = 0$ we have $s = 0$. So we try $x_p(t) = t^0(A_2 t^2 + A_1 t + A_0)e^{0t} = A_2 t^2 + A_1 t + A_0$. Substituting this into $2x' + x = 3t^2$ gives

$$2(2A_2 t + A_1) + (A_2 t^2 + A_1 t + A_0) = 3t^2.$$

Matching coefficients of powers gives these three equations:

$$\begin{aligned} A_2 &= 3, \\ 4A_2 + A_1 &= 0, \\ 2A_1 + A_0 &= 0. \end{aligned}$$

These are easy to solve, in the given order for instance, to give $A_0 = 24$, $A_1 = -12$, $A_2 = 3$. In fact it is easy to check that $x_p(t) = 3t^2 - 12t + 24$ is a solution of $2x' + x = 3t^2$.

4.4 #16. The right side has form $Ct^m e^{\alpha t} \sin(\beta t) = t \sin t$ so $m = 0$, $\alpha = 0$, $\beta = 1$. The issue is whether $r = \alpha \pm i\beta = \pm i$ are roots of the auxiliary equation, which is $r^2 - 1 = 0$. But $r = \pm i$ does *not* solve $r^2 - 1 = 0$. So $s = 0$ and we try this form

$$\theta_p(t) = t^0(A_1 t + A_0)e^{0t}\cos(1t) + t^0(B_1 t + B_0)e^{0t}\sin(1t) = (A_1 t + A_0)\cos t + (B_1 t + B_0)\sin t.$$

Substitution into $\theta'' - \theta = t \sin t$, and simplification, gives

$$(-2A_1 t - 2A_0 + 2B_1)\cos t + (-2B_1 t - 2A_1 - 2B_0)\sin t = t \sin t.$$

The coefficients must match:

$$\begin{aligned} -2A_1 &= 0, \\ -2A_0 + 2B_1 &= 0, \\ -2B_1 &= 1, \\ -2A_1 - 2B_0 &= 0. \end{aligned}$$

As a (checkable!) result,

$$\theta_p(t) = -\frac{1}{2}\cos t - \frac{1}{2} t \sin t.$$

4.4 #22. The right side has form $24t^2e^t = Ct^me^{rt}$ so $m = 2$ and $r = 1$. The auxiliary equation is $r^2 - 2r + 1 = 0$ and $r = 1$ (appearing on the right side) *is* a root. Indeed $r^2 - 2r + 1 = (r-1)^2$ so $r = 1$ is a repeated root, and thus $s = 2$. So we try this form

$$x_p(t) = t^2(A_2t^2 + A_1t + A_0)e^t.$$

Substitution of this form into $x'' - 2x' + x = 24t^2e^t$, and a substantial amount of work (!), gives these three easy equations for A_2, A_1, A_0, by matching coefficients of like powers; note that the highest powers of t have coefficient zero: $2A_0 = 0$, $6A_1 = 0$, $12A_2 = 24$. Thus

$$x_p(t) = t^2(2t^2 + 0t + 0)e^t = 2t^4e^t.$$

This is checkable, worth checking, and checks out!

4.4 #28. (*Note only the* form *of $y_p(t)$ is asked for.*) The right side (nonhomogeneity) has form $t^4e^t = Ct^me^{rt}$ for $m = 4$ and $r = 1$. The auxiliary equation is $r^2 + 3r - 7 = 0$. Note $1^2 + 3(1) - 7 \neq 0$ so $s = 0$. Thus we should try this form:

$$y_p(t) = \left(A_4t^4 + A_3t^3 + A_2t^2 + A_1t + A_0\right)e^t.$$

4.7 #10. Substituting t^r gives characteristic equation

$$r(r-1) + 2r - 6 = r^2 - r - 6 = 0.$$

That is, $(r-3)(r+2) = 0$, so the general solution is

$$y(t) = c_1t^3 + c_2t^{-2}.$$

At least that was easy ...

4.7 #46. Here $y_1(t) = t^{-2}$ is given. The other thing we need for reduction of order is $p(t)$. But the equation must be in standard form to know $p(t)$:

$$y'' + \frac{6}{t}y' + \frac{6}{t^2}y = 0.$$

Thus $p(t) = 6/t$. Reduction of order is, therefore, this nested pair of integrals:

$$\begin{aligned} y_2(t) &= y_1 \int \frac{e^{-\int p\,dt}}{y_1^2}\,dt = t^{-2}\int \frac{e^{-\int (6/t)dt}}{t^{-4}}\,dt = t^{-2}\int t^4e^{-6\ln t}\,dt \\ &= t^{-2}\int t^4t^{-6}\,dt = t^{-2}\int t^{-2}\,dt = -t^{-3}. \end{aligned}$$

I have done these integrals quickly, ignoring constants, because we are only looking for one new solution y_2. The general solution $c_1y_1 + c_2y_2$ will have unknown constants anyway.

In this case we can check the answer two ways. First we may substitute $y_2 = -t^{-3}$ directly to see it is a solution. Second we can notice the ODE is actually a Cauchy-Euler equation.