Selected Solutions to Assignment #5

(Revised 4.3 #4.)

These problems were graded at 3 points each for a total of 27 points.

4.1 #2. **(a)** $m(cy)'' + b(cy)' + k(cy) = c(my'' + by' + ky) = c(0) = 0$

(b)

$$\begin{aligned} m(y_1 + y_2)'' + b(y_1 + y_2)' + k(y_1 + y_2) &= my_1'' + my_2'' + by_1' + by_2' + ky_1 + ky_2 \\ &= (my_1'' + by_1' + ky_1) + (my_2'' + by_2' + ky_2) = 0 + 0 = 0 \end{aligned}$$

4.1 #8. Substituting $y(t) = A\sin 3t + B\cos 3t$ into the differential equation gives

$$(-9A\sin 3t - 9B\cos 3t) + 2(3A\cos 3t - 3B\sin 3t) + 4(A\sin 3t + B\cos 3t) = \sin 3t$$

or

$$(-9B + 6A + 4B)\cos 3t + (-9A - 6B + 4A)\sin 3t = \sin 3t$$

or

$$(6A - 5B)\cos 3t + (-5A - 6B)\sin 3t = 0\ \cos 3t + 1\sin 3t.$$

These expressions are to be equal for all t, so the coefficients of $\sin 3t$ and $\cos 3t$ must be equal, respectively:

$$\begin{aligned} 6A - 5B &= 0 \\ -5A - 6B &= 1 \end{aligned}$$

The solution is $A = -5/61$, $B = -6/61$.

4.2 #4. The auxiliary (characteristic) equation is $r^2 - r - 2 = 0$, which you get by substituting $y(t) = e^{rt}$ into the equation, using the chain rule, and then dividing by $y(t) = e^{rt}$. The characteristic equation factors, $(r-2)(r+1) = 0$, so $r_1 = 2$ and $r_2 = -1$ are the distinct real roots. The general solution is

$$y(t) = c_1 e^{2t} + c_2 e^{-t}.$$

4.2 #14. Here the characteristic equation is $r^2 + r = 0$ so $r_1 = 0$ and $r_2 = -1$ are roots, so

$$y(t) = c_1 e^{0t} + c_2 e^{-t} = c_1 + c_2 e^{-t}.$$

(*Notice that constant functions actually are solutions of the differential equation.*) The initial values imply

$$\begin{aligned} 2 &= y(0) = c_1 + c_2 \\ 1 &= y'(0) = -c_2 \end{aligned}$$

Therefore $c_2 = -1$, $c_1 = 3$, and the solution of the initial value problem is

$$y(t) = 3 - e^{-t}.$$

(*And this is easy to check.*)

4.2 #22. The characteristic equation is $3r - 7 = 0$, so $r = 7/3$ is the (only) root. The general solution is $y(t) = Ce^{(7/3)t}$.

4.3 #4. *REVISED.* Here the auxiliary (characteristic) equation is $r^2 - 10r + 26 = 0$ with roots $r = (10 \pm \sqrt{10^2 - 4(26)})/2 = 5 \pm i$. Therefore the general solution is

$$y(t) = e^{5t}\left(c_1\cos(t) + c_2\sin(t)\right).$$

4.3 #18. Here the characteristic equation is $2r^2+13r-7=0$ with roots $r = (-13 \pm \sqrt{13^2 - 4(2)(-7)})/4 = -(13/4) \pm (15/4) = -7, 1/2$. Therefore the general solution is

$$y(t) = c_1 e^{-7t} + c_2 e^{t/2}.$$

4.3 #24. Again, the characteristic equation is $r^2 + 9 = 0$ with roots $r = \pm 3i$, so the general solution is

$$y(t) = c_1 \cos(3t) + c_2 \sin(3t).$$

The initial conditions state

$$1 = y(0) = c_1, \qquad 1 = y'(0) = 3c_2.$$

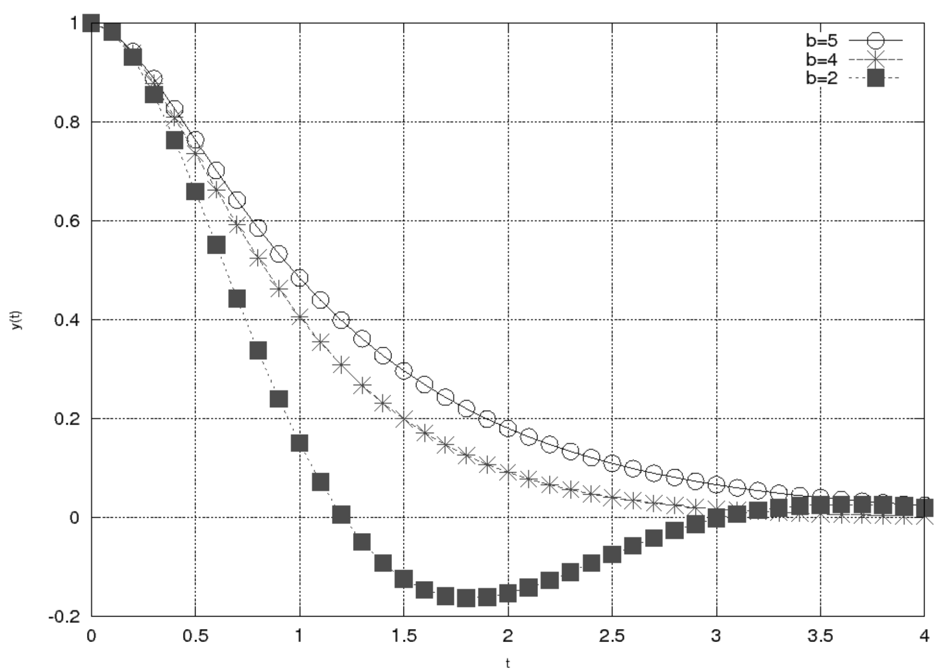
The solution to the initial value problem is

$$y(t) = \cos(3t) + \frac{1}{3}\sin(3t).$$

4.3 #28. Leaving the "b" in place, the characteristic equation is $r^2 + br + 4 = 0$ with roots $r = (-b \pm \sqrt{b^2 - 16})/2$. There are three cases:

- $b = 5$: Roots: $r = -4, -1$. General solution: $y(t) = c_1 e^{-4t} + c_2 e^{-t}$, $y'(t) = -4c_1 e^{-4t} - c_2 e^{-t}$. Initial conditions: $1 = y(0) = c_1 + c_2$, $0 = y'(0) = -4c_1 - c_2$. Solution to initial value problem: $y(t) = -(1/3)e^{-4t} + (4/3)e^{-t}$.
- $b = 4$: Roots: $r = -2$ (repeated). General solution: $y(t) = e^{-2t}(c_1 + c_2 t)$, $y'(t) = e^{-2t}(-2c_1 + (1-2t)c_2)$. Initial conditions: $1 = y(0) = c_1$, $0 = y'(0) = -2c_1 + c_2$. Solution to initial value problem: $y(t) = e^{-2t}(1 + 2t)$.
- $b = 2$: Roots: $r = -1 \pm \sqrt{3}\, i$. General solution: $y(t) = e^{-t}\left(c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t\right)$, $y'(t) = e^{-t}\left((-c_1 + \sqrt{3}c_2)\cos\sqrt{3}t + (-\sqrt{3}c_1 + c_2)\sin\sqrt{3}t\right)$. Initial conditions: $1 = y(0) = c_1$, $0 = y'(0) = -c_1 + \sqrt{3}c_2$. Solution to initial value problem: $y(t) = e^{-t}\left(\cos\sqrt{3}t + (1/\sqrt{3})\sin\sqrt{3}t\right)$.

Plotting these gives:



We see that all curves start at the same location with the same slope, as they should. Further the effect of the coefficient "b" is clear, think in terms of the mass-spring analogy. Smaller values of b mean less damping. The case $b = 4$ happens to be critical damping, which we know because the root is repeated.

We can expect that when $b = 0$ the curve starts reasonably closely to the $b = 2$ but enters an oscillation of constant amplitude. (*And you can easily solve the* $b = 0$ *case, right?*)

By the way, here's the MATLAB/OCTAVE code which produced the figure:

```
t = 0:.1:4;
y5 = -(1/3)*exp(-4*t) + (4/3)*exp(-t);
y4 = exp(-2*t) .* (1+2*t);
y2 = exp(-t) .* ( cos(sqrt(3)*t) + (1/sqrt(3))*sin(sqrt(3)*t) );
plot(t,y5,'o-','markersize',12,...
     t,y4,'*-','markersize',12,...
     t,y2,'s-','markersize',12)
legend("b=5","b=4","b=2")
xlabel t, ylabel('y(t)')
grid on
```