

Solutions to Assignment #4

These problems were graded at 3 points each for a total of 21 points.

3.4 #2. This problem fits into the framework of Example 1. In particular, we are told $b = 10$ and $g = 32$. But note $mg = 400$ so $m = 400/32$; *pounds are a unit of force not mass*. Let $x(t)$ be the distance the object has fallen, in feet, and $v(t)$ be the velocity $v = dx/dt$, so v is positive if the object is falling.

Then $m(dv/dt) = mg - bv$ and we can use formula (6) on page 117:

$$x(t) = \frac{400}{10}t + \frac{400}{32(10)} \left(0 - \frac{400}{10}\right) \left(1 - e^{-(320/400)t}\right) = 40t - 50 \left(1 - e^{-(4/5)t}\right).$$

We seek the time when the object has fallen 500 ft:

$$500 = x(t) = 40t - 50 \left(1 - e^{-(4/5)t}\right).$$

This equation cannot be solved exactly. But t must be of size ≈ 10 seconds because the term in parentheses in the last equation is between 0 and 1. That means “ $e^{-(4/5)t}$ ” must be very small at the time of impact. So we ignore that exponential part and solve $500 = 40t - 50$ so $t = 550/40 = 13.75$ seconds.

3.4 #8. Again this problem fits in the framework of Example 1, with $m = 100$ kg, $g = 9.81$ m/s², and b given according to whether the parachute is open or closed. Remember that $x(t)$ is the distance fallen in t seconds, measured in meters, and that v is positive when falling.

If the chute opens at 30 seconds then for the first 30 seconds $b = 20$,

$$x(t) = \frac{100(9.81)}{20}t + \frac{100}{20} \left(0 - \frac{100(9.81)}{20}\right) \left(1 - e^{-(20/100)t}\right) = \frac{981}{20}t - \frac{981}{4} \left(1 - e^{-(1/5)t}\right),$$

$$v(t) = \frac{100(9.81)}{20} + \left(0 - \frac{100(9.81)}{20}\right) e^{-(20/100)t} = \frac{981}{20} \left(1 - e^{-(1/5)t}\right).$$

So we can find the distance fallen and the velocity at the time the chute opens, by noting that when $t = 30$ seconds we have $x(30) = 1226.86$ m and $v(30) = 48.93$ m/s. This gives a new problem with a new value for b and for v_0 :

$$x(t) = \frac{100(9.81)}{100}t + \frac{100}{100} \left(48.93 - \frac{100(9.81)}{100}\right) \left(1 - e^{-(100/100)t}\right)$$

Now we only need the additional time so that the distance fallen is $3000 - 1226.86 = 1773.14$ m. Setting $x(t) = 1773.14$ in the last equation we get $t \approx 176.76$ seconds, because t is large enough so that the e^{-t} term is negligible. Thus the parachutist hits the ground $176.76 + 30 = 206.76$ seconds after jumping.

The case where the chute opens after 1 minute = 60 seconds, the result is similar, but with $x(60) = 2697.75$ m, $v(60) = 49.05$ m/s, and finally we find that the parachutist hits the ground after 86.81 seconds. Thus there is substantially less total time falling if the chute opens later, of course, with the least time taken when the chute does not open at all ...

3.4 #18. Here the force is constant. In particular, the force of friction, *kinetic* friction, is independent of the speed at which the object slides. Also, the magnitude of the normal force applied by the object to the inclined plane is exactly the component of the force of gravity which is perpendicular to the plane, i.e. $mg \cos 30^\circ$. In particular,

$$m \frac{dv}{dt} = F = mg \sin 30^\circ - \mu N = mg \sin 30^\circ - \mu mg \cos 30^\circ = mg (\sin 30^\circ - \mu \cos 30^\circ).$$

We have values for all constants. Notice that the mass of the object is irrelevant:

$$\frac{dv}{dt} = 9.81 \left(\frac{1}{2} - 0.2 \frac{\sqrt{3}}{2} \right) = 3.206 \frac{\text{m}}{\text{s}^2}.$$

That is, the acceleration is about 1/3 of gravity. Since $v(0) = 0$ we have $v(t) = 3.206 t$ so $x(t) = 1.603 t^2$ because $x(0) = 0$.

Finally we can answer the question. We find t when $x(t) = 5$ m. The result is $t = 1.766$ seconds, at which time the velocity is $v(t) = 5.66$ m/s.

Incidentally, if we consider inclined planes with angle “ θ ” in place of “ 30° ”, we see that there is no force on the object when

$$\sin \theta - \mu \cos \theta = 0.$$

This gives a meaning to μ , that is, $\mu = \tan \theta$ where θ is the “friction angle”, the angle at which kinetic friction balances gravity. When $\mu = 0.2$ the corresponding friction angle is 11.3° , so sliding occurs on any inclined plane steeper than that. As we see.

3.5 #1. See back of text for answer.

3.5 #2. This question was addressed briefly in class. See equation (4) on page 127. With the given constants it reduces to

$$\frac{dq}{dt} + 10^6 q = \sin(100t).$$

This equation is linear. The solution is given by

$$q(t) = e^{-10^6 t} \int e^{10^6 t} \sin(100t) dt.$$

Integration by parts gives

$$\int e^{10^6 t} \sin(100t) dt = \frac{1}{10^8 + 1} e^{10^6 t} (10^2 \sin(100t) - 10^{-2} \cos(100t)) + C,$$

so

$$q(t) = \frac{1}{10^8 + 1} (10^2 \sin(100t) - 10^{-2} \cos(100t)) + C e^{-10^6 t}.$$

Because the initial capacitor voltage is zero, the initial charge $q(0) = 0$, so $C = (10^{10} + 10^2)^{-1}$.

We don't want q itself but rather the capacitor voltage q/C and the resistor voltage $E(t) - (q/C)$ and the current $I = dq/dt$:

$$(\text{capacitor voltage}) = \frac{q}{C} = 10^4 \sin(100t) - \cos(100t) + e^{-10^6 t}.$$

$$(\text{resistor voltage}) = E(t) - \frac{q}{C} = (1 - 10^4) \sin(100t) + \cos(100t) - e^{-10^6 t},$$

$$I(t) = \frac{dq}{dt} = \frac{1}{10^8 + 1} (10^4 \cos(100t) + \sin(100t)) - \frac{10^4}{10^8 + 1} e^{-10^6 t}.$$

3.6 #2. Euler's method in this case says

$$y_{n+1} = y_n + h \left(-\frac{1}{2} y_n \right) = \left(1 - \frac{h}{2} \right) y_n.$$

Because of the initial condition $y_0 = 3$. Thus $y_1 = (1 - (h/2)) 3$, $y_2 = (1 - (h/2))(1 - (h/2)) 3 = (1 - (h/2))^2 3$, and so on.

Now, we seek y_n which approximates $y(2)$. There is a relationship between the number of steps n and the stepsize h :

$$n h = 2.$$

Thus

$$y(2) \approx y_{n=2/h} = \left(1 - \frac{h}{2}\right)^n 3 = \left(1 - \frac{h}{2}\right)^{2/h} 3$$

3.6 #8. I wrote the following program. Note that the equation is separable so it is possible to compare Improved Euler to Euler to the exact solution.

```
% solve exercise 3.6 #8 using improved Euler:
% y' = (1/x) (y^2 + y), y(1) = 1

h = 0.2;
x = [1.0 1.2 1.4 1.6 1.8];
y = [1.0 0 0 0 0]; % space for solution, and set initial condition
yE = y;

for n=1:4
    % see equation (9) in 3.6:
    fn = (1.0/x(n)) * (y(n)^2 + y(n));
    ynp = y(n) + h * fn;
    fnp = (1.0/x(n+1)) * (ynp^2 + ynp);
    y(n+1) = y(n) + (h/2) * (fn + fnp);

    % do Euler for comparison
    yE(n+1) = yE(n) + h * (1.0/x(n)) * (yE(n)^2 + yE(n));
end

yexact = x./(2-x); % equation is separable, so compare to exact

plot(x,y,'o-',x,yE,'*-',x,x./(2-x),'x-')
legend("improved Euler","Euler","exact")

x,y,yE,yexact % just show numbers
```

Here's the text output:

```
>> exer3pt6_8
x =
    1.0000    1.2000    1.4000    1.6000    1.8000
y =
    1.0000    1.4800    2.2478    3.6517    6.8871
yE =
    1.0000    1.4000    1.9600    2.7888    4.1096
yexact =
    1.0000    1.5000    2.3333    4.0000    9.0000
```