

Selected Solutions to Assignment #3

These problems were graded at 3 points each for a total of 27 points.

2.4 #6. separable, linear (and not exact)

2.4 #10. It is exact because

$$1 = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1.$$

Because $M = \partial F/\partial x$,

$$F(x, y) = \int (2x + y) dx + g(y) = x^2 + yx + g(y).$$

On the other hand, $N = \partial F/\partial y$ so

$$x - 2y = N = \frac{\partial F}{\partial y} = x + g'(y),$$

which says $g'(y) = -2y$. But then $g(y) = -y^2$; we can take the constant to be zero in this antiderivative. We now know an F so that $M = \partial F/\partial x$ and $N = \partial F/\partial y$. The solution of the differential equation is a level curve of this F :

$$C = F(x, y) = x^2 + yx - y^2.$$

This can be solved for y by the quadratic formula, but that hardly seems helpful. More helpful, perhaps, is to notice that the solutions are conic sections, and hyperbolas in particular.

2.4 #15. See back of text for answer.

2.4 #22. Check for exactness:

$$e^{xy} + ye^{xy}x + \frac{1}{y^2} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = e^{xy} + xe^{xy}y + \frac{1}{y^2},$$

passed. From $M = \partial F/\partial x$,

$$F(x, y) = \int ye^{xy} - \frac{1}{y} dx + g(y) = e^{xy} - \frac{x}{y} + g(y).$$

On the other hand, $N = \partial F/\partial y$ so

$$xe^{xy} + \frac{x}{y^2} = N = \frac{\partial F}{\partial y} = xe^{xy} + \frac{x}{y^2} + g'(y),$$

which says $g'(y) = 0$. Choose $g(y) = 0$ for simplicity. Then F is known and the solution is a level curve of it:

$$C = F(x, y) = e^{xy} - \frac{x}{y}.$$

Finally our initial value $y(1) = 1$ chooses C :

$$C = e^{1 \cdot 1} - \frac{1}{1} = e - 1,$$

so the implicit solution is

$$e^{xy} - \frac{x}{y} = e - 1.$$

(I can't solve for y or x . But I can draw a contour plot of $z = F(x, y)$, and pick the $z = e - 1$ contour!)

2.4 #28. The test for exactness is $\partial M/\partial y = \partial N/\partial x$. Here we are given M and seek N . So we write down the test for exactness and treat it as a differential equation for N . We integrate with respect to x , noting that if an arbitrary function of y is added to N then the test for exactness is unaffected:

(a)

$$\cos(xy) - xy \sin(xy) = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Integrate to find the general form of N , using the substitution $u = xy$, $du = y dx$:

$$\begin{aligned} N(x, y) &= \int \cos(xy) - xy \sin(xy) dx + g(y) = \frac{1}{y} \int \cos(u) - u \sin(u) du + g(y) \\ &= \frac{1}{y} (u \cos(u)) + g(y) = x \cos(xy) + g(y). \end{aligned}$$

(b) Very similar, even including the same u -substitution:

$$e^{xy} + xye^{xy} - 4x^3 = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

$$\begin{aligned} N(x, y) &= \int e^{xy} + xye^{xy} - 4x^3 dx + g(y) = \frac{1}{y} \left(\int e^u + ue^u du \right) - x^4 + g(y) \\ &= \frac{1}{y} (ue^u) - x^4 + g(y) = xe^{xy} - x^4 + g(y). \end{aligned}$$

2.6 #13. See back of text for answer.

2.6 #14. Rewrite as

$$\frac{dy}{d\theta} = \sec\left(\frac{y}{\theta}\right) + \frac{y}{\theta} = \sec v + v$$

where $v = y/\theta$. Since $\theta v = y$, the ODE is

$$v + \theta \frac{dv}{d\theta} = \sec v + v \quad \text{or} \quad \theta \frac{dv}{d\theta} = \sec v,$$

which is separable. Separating variables and integrating,

$$\int \cos v dv = \int \frac{d\theta}{\theta} \quad \text{or} \quad \sin v = \ln |\theta| + c.$$

Thus $v(\theta) = \arcsin(\ln |\theta| + c)$. But we seek $y(\theta)$, so

$$y(\theta) = \theta v(\theta) = \theta \arcsin(\ln |\theta| + c).$$

3.2 #2. By the standard reasoning, if $x(t)$ is the number of kg of salt in the tank at time t then

$$\frac{dx}{dt} = (\text{input of salt per time}) - (\text{output of salt per time}) = 6(0.05) - 6\left(\frac{x(t)}{50}\right),$$

$$\frac{dx}{dt} + 0.12x(t) = 0.30.$$

Notice that both sides of each form have units of kg per min. The initial value is $x(0) = 0.5$ kg. The equation is linear. The general solution involves $\mu(t) = e^{0.12t}$, and gives

$$\mu(t)x(t) = \int \mu(t)0.30 dt = 0.30 \int e^{0.12t} dt = (0.30)(50/6)e^{0.12t} + c = 2.5e^{0.12t} + c,$$

$$x(t) = \mu^{-1} (2.5e^{0.12t} + c) = 2.5 + ce^{-0.12t}.$$

On the other hand, $x(0) = 0.5$ so $c = -2$ and we have $x(t) = 2.5 - 2e^{-0.12t}$. Finally we can answer the question. Seek t so that $(0.03)50 = 1.5 = 2.5 - 2e^{-0.12t}$ or $t = -\ln(1/2)/(0.12) = 5.78$ min.

3.2 #18. The values from the table are, if 1900 is $t = 0$ and we measure t in years and U.S. population in millions, $p(0) = p_0 = 76.21$, $p(20) = 106.02$, and $p(40) = 132.16$. To fit the logistic model, equation (15) on page 101, it remains to find p_1 and A . From the data we have two equations, for 1920 and 1940 respectively:

$$106.02 = \frac{76.21p_1}{76.21 + (p_1 - 76.21)e^{-20Ap_1}},$$

$$132.16 = \frac{76.21p_1}{76.21 + (p_1 - 76.21)e^{-40Ap_1}}.$$

As described in the text, the result of problem 12 finds p_1 and A , where $p_a = 106.02$, $p_b = 132.16$, and $t_a = 20$:

$$p_1 = \left[\frac{(106.02)(132.16) - 2(76.21)(132.16) + (76.21)(106.02)}{(106.02)^2 - (76.21)(132.16)} \right] (106.02) = 176.73,$$

$$A = \frac{1}{(176.73)(20)} \ln \left[\frac{(132.16)(106.02 - 76.21)}{(76.21)(132.16 - 106.02)} \right] = 1.929 \times 10^{-4} = 0.0001929.$$

Note that the “limiting population” p_1 is much smaller than the current U.S. population. The reason is that the 20 years from 1900 to 1920 saw a larger population increase (30 million) than the following 20 years (26 million), so the logistic model “thinks” that the S-curve, like the left-hand part of Figure 3.4 in the text, is already “kicking in”, and that the population will soon level out. In fact, in the 20 years after 1940 the population increased by 47 million. We see here the difference between immigration both before and after World War I, versus birth rates in the Great Depression and in the postwar prosperity and baby-boom. The logistic model, or any other rational model, does not capture human history.

See Figure 3.5 for a suggestion of how to improve the model somewhat. We should be fitting a logistic model to many data points using a least squares method, for example.

Continuing to make predictions, the logistic model, (15) on page 101, says that in 1990 ($t = 90$) we have $p(90) = 166.52$ and in 2000 ($t = 100$) we have $p(100) = 169.34$. Both of these are lower than $p_1 = 176.73$ million, the limiting population predicted by the logistic model. They are much lower than census values of 248.71 and 281.42 million, respectively.