Selected Solutions to Assignment #2

Revised. Corrections in 2.2 #12 and 2.3 #8.

1.4 #8. The full table of my Euler's method calculations looks like this:

steps	x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
1	y	0								π
2	y	0				$\frac{\pi}{2} = 1.5708$				$\frac{\pi}{2} = 1.5708$
4	y	0		0.78540		1.0154		1.1335		1.2074
8	y	0	0.39270	0.63512	0.79484	0.90725	0.99058	1.0548	1.1060	1.1476

The approximations to $y(\pi)$ are 3.1416, 1.5708, 1.2074, and 1.1476 for $1, 2, 4, 8$ steps, respectively.

(*By the way, with* 1000 *steps I get* $y(\pi) \approx 1.0973$ *and with* $100,000$ *steps* $y(\pi) \approx 1.0970$. *I don't know an easy way to solve the ODE* $y' = 1 - \sin y$ *exactly, though it is separable.*)

2.2 #4. Because the right side can be manipulated into the desired factored form,

$$\frac{ds}{dt} = t \ln\left(s^{2t}\right) + 8t^2 = t^2\left(2\ln s + 8\right),$$

the equation is separable.

2.2 #8. Separate and integrate:

$$\int y^3\,dy = \int \frac{dx}{x}, \qquad \frac{1}{4}y^4 = \ln|x| + c, \qquad y = \left(4\ln|x| + C\right)^{1/4}.$$

2.2 #12. Separate and integrate using (e.g.) substitution $u = 1 - 4v^2$:

$$\begin{aligned}
\int \frac{3v\,dv}{1-4v^2} &= \int \frac{dx}{x}, \\
-\frac{3}{8}\int \frac{du}{u} &= \ln|x| + c, \\
-\frac{3}{8}\ln|1-4v^2| &= \ln|x| + c, \\
|1-4v^2|^{-3/8} &= A|x|, \\
1-4v^2 &= B|x|^{-8/3}, \\
v(x) &= \pm\frac{1}{2}\left(1 - B|x|^{-8/3}\right)^{1/2}.
\end{aligned}$$

2.2 #22. Separate and integrate:

$$\int 2y\,dy = \int -x^2\,dx, \qquad y^2 = -\frac{1}{3}x^3 + c.$$

This is a good stage to find c from the initial condition: $2^2 = 0 + c$ so $c = 4$. Then

$$y(x) = \left(4 - \frac{1}{3}x^3\right)^{1/2}.$$

Use the positive square root because $y(0) = 2$ is positive. (I.e. evaluating at $x = 0$ gives $+2$.)

2.3 #8. Linear. In standard form "$\frac{dy}{dx} + P(x)y = Q(x)$" it is $\frac{dy}{dx} - (1/x)y = 2x + 1$ so $P = -(1/x)$ and $\mu = e^{\int P} = e^{-\ln|x|} = |x|^{-1}$. The stage "$\frac{d}{dx}(\mu y) = \mu Q$" is $\frac{d}{dx}\left(|x|^{-1}y\right) = |x|^{-1}(2x+1)$ but we can multiply both sides by either $+1$ or -1 so the absolute values can be removed. So the rest of the calculation goes like this:

$$\frac{d}{dx}\left(x^{-1}y\right) = \frac{2x+1}{x} = 2 + \frac{1}{x},$$
$$\frac{y}{x} = \int 2 + \frac{1}{x}\, dx = 2x + \ln|x| + c, \qquad y(x) = x(2x + \ln|x| + c).$$

2.3 #18. Linear. $\mu = e^{4x}$. Starting from the stage "$\frac{d}{dx}(\mu y) = \mu Q$":

$$\frac{d}{dx}\left(e^{4x}y\right) = e^{4x}e^{-x} = e^{3x},$$
$$e^{4x}y = \int e^{3x}\, dx = \frac{1}{3}e^{3x} + c,$$
$$y(x) = \frac{1}{3}e^{-x} + c\, e^{-4x}.$$

But $y(0) = 4/3$ so $c = 1$ and $y(x) = (1/3)e^{-x} + e^{-4x}$.

2.3 #30. **(a)** If $v = y^3$ then

$$\frac{dv}{dx} = 3y^2\frac{dy}{dx}.$$

On the other hand, multiplying the original equation by y^2 gives

$$y^2\frac{dy}{dx} + 2y^3 = x.$$

This means we know how to replace all quantities containing "y" with corresponding expressions in terms of "v":

$$\frac{1}{3}\frac{dv}{dx} + 2v = x$$

or $\frac{dv}{dx} + 6v = 3x$.

(b) The last equation is linear. $\mu(x) = e^{6x}$. The next steps look like this:

$$\frac{d}{dx}\left(e^{6x}v\right) = 3xe^{6x},$$
$$e^{6x}v = 3\int xe^{6x}\, dx = 3\left(\frac{1}{6}xe^{6x} - \frac{1}{36}e^{6x} + c\right),$$
$$v = \frac{1}{2}x - \frac{1}{12} + 3c\, e^{-6x} = \frac{1}{2}x - \frac{1}{12} + Ce^{-6x},$$
$$y^3 = \frac{1}{2}x - \frac{1}{12} + Ce^{-6x},$$
$$y(x) = \left(\frac{1}{2}x - \frac{1}{12} + Ce^{-6x}\right)^{1/3}.$$

The integral was done "by parts". The last formula, for the general solution, can be checked by substituting into the original equation.