Selected Solutions to Assignment #10

These problems were graded at 5 points each for a total of 25 points.

7.5 #4. The Laplace transform of the ODE, including the initial values, is

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 6\left[sY(s) - y(0)\right] + 5Y(s) = 12\frac{1}{s-1},$$
$$(s^2+6s+5)Y(s) = \frac{12}{s-1} - s + 1,$$
$$Y(s) = \frac{\frac{12}{s-1} - s + 1}{s^2+6s+5} = \frac{-s^2+2s+11}{(s-1)(s+1)(s+5)}.$$

Partial fractions gives

$$Y(s) = \frac{-s^2+2s+11}{(s-1)(s+1)(s+5)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+5}$$
$$-s^2+2s+11 = A(s+1)(s+5) + B(s-1)(s+5) + C(s-1)(s+1)$$

Substitution of $s=1$, $s=-1$, $s=-5$ gives $A=1$, $B=-1$, $C=-1$, respectively. Thus

$$Y(s) = \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{s+5}$$
$$y(t) = e^t - e^{-t} - e^{-5t}.$$

It *actually is* easy to check that this formula gives the solution $y(t)$ of the initial value problem.

7.5 #16. (*This problem is easy because you are not asked to invert the Laplace transform.*) Applying the Laplace transform to both sides of the equation,

$$s^2Y(s) - sy(0) - y'(0) + 6Y(s) = \frac{2}{s^3} - \frac{1}{s} \quad\Longleftrightarrow\quad (s^2+6)Y(s) = \frac{2}{s^3} - \frac{1}{s} - 1$$
$$Y(s) = \frac{\frac{2}{s^3} - \frac{1}{s} - 1}{s^2+6} = \frac{2-s^2-s^3}{s^3(s^2+6)}.$$

7.5 #26. (*This one was carefully designed to make the numbers simple.*) Apply the Laplace transform and simplify:

$$\left[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right] + 4\left[s^2Y(s) - sy(0) - y'(0)\right] + \left[sY(s) - y(0)\right] - 6Y(s) = \frac{-12}{s}$$
$$(s^3+4s^2+s-6)Y(s) = s^2+8s+15 - \frac{12}{s}$$
$$Y(s) = \frac{s^3+8s^2+15s-12}{s(s^3+4s^2+s-6)} = \frac{s^3+8s^2+15s-12}{(s+3)(s+2)(s)(s-1)}.$$

At the last stage we have had to factor a cubic: $s^3+4s^2+s-6 = (s+3)(s+2)(s-1)$. This is inconvenient.[1] It is efficient to factor by machine. Here is the MATLAB:

[1] It is also unavoidable because the same polynomial must be factored by the previous methods, which gave auxiliary equation "$r^3+4r^2+r-6=0$".

```
>> help roots              % so that I can recall how "roots" works
>> roots([1 4 1 -6])
ans =
  -3.0000
  -2.0000
   1.0000
```

Proceeding, we do partial fractions:

$$Y(s) = \frac{s^3 + 8s^2 + 15s - 12}{(s+3)(s+2)(s)(s-1)} = \frac{A}{s+3} + \frac{B}{s+2} + \frac{C}{s} + \frac{D}{s-1}.$$

Clearing denominators and substituting $s = -3, -2, 0, 1$ gives $A = 1$, $B = -3$, $C = 2$, $D = 1$, respectively. The inverse transform is easy, giving

$$y(t) = e^{-3t} - 3e^{-2t} + 2 + e^t.$$

The initial conditions and the ODE itself are all easy to check. This is the right answer!

8.1 #2. We will use the values $y(0), y'(0), y''(0), \dots$ to determine the coefficients of the Taylor polynomial at $x_0 = 0$. We differentiate the ODE and plug in $x = 0$. We stop once we have three nonzero coefficients including $y(0) = 2$:

$$y' = y^2 \quad \implies \quad y'(0) = y(0)^2 = 4$$
$$y'' = 2y'y \quad \implies \quad y''(0) = 2y'(0)y(0) = 16.$$

Thus the quadratic Taylor polynomial approximation which has the correct first three terms is

$$p_2(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 = 2 + 4x + 8x^2.$$

Optional. We can also solve exactly because the ODE is first-order and separable. The exact solution is $y(x) = 2/(1-2x)$. The Taylor series for the exact solution is

$$y(x) = \frac{2}{1-2x} = 2 + 4x + 8x^2 + \dots$$

So we got the right first three coefficients.

8.1 #6. The method is essentially the same as for **#2**, but using both initial values ($y(0) = 0$, $y'(0) = 1$) to get started:

$$y'' = -y \quad \implies \quad y''(0) = -y(0) = 0$$
$$y''' = -y' \quad \implies \quad y'''(0) = -y'(0) = -1$$
$$y^{(4)} = -y'' \quad \implies \quad y^{(4)}(0) = -y''(0) = 0$$
$$y^{(5)} = -y''' \quad \implies \quad y^{(5)}(0) = -y'''(0) = +1.$$

Again we have stopped once we have three nonzero coefficients. The quintic Taylor polynomial which has the correct first three nonzero coefficients is

$$p_5(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5.$$

Optional. We can also solve exactly because the ODE is constant-coefficient and linear and homogeneous. In fact it is very quick to see that the exact solution is

$$y(x) = \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$