Math 252 Calculus II (Bueler) 2 February 2018

10 Improper Integrals: Solutions

A. TYPE 2

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^-}\int_t^1 x^{1/2}\,dx = \lim_{t\to 0^-}\left[2x^{1/2}\right]_t^1 = 2\lim_{t\to 0^-}\left[1-t^{1/2}\right] = 2$$

B. Using $u = x^2$: TYPE 1

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx = \lim_{t\to\infty}\int_0^{t^2} e^{-u}\frac{du}{2} = \frac{1}{2}\lim_{t\to\infty}\left[1-e^{-t^2}\right] = \frac{1}{2}$$

C. DIVERGES, TYPE 1

$$\int_{-\infty}^{-1}\frac{dx}{x} = \lim_{t\to-\infty}\int_t^{-1}\frac{dx}{x} = \lim_{t\to-\infty}\Big[\ln|x|\Big]_t^{-1} = \lim_{t\to-\infty}\ln 1 - \ln(-t) = -\infty$$

D. DIVERGES, TYPE 1

$$\int_0^\infty e^x\,dx = \lim_{t\to\infty}\int_0^t e^x\,dx = \lim_{t\to\infty}\Big[e^x\Big]_0^t = \lim_{t\to\infty} e^t - 1 = +\infty$$

E. Use $u = 5 - x$ to clarify which limit is improper: TYPE 2

$$\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,dx = \int_5^0 u^{-1/3}\,(-du) = \int_0^5 u^{-1/3}\,du = \lim_{t\to 0^+}\int_t^5 u^{-1/3}\,du = \lim_{t\to 0^+}\left[\frac{3}{2}u^{2/3}\right]_t^5$$
$$= \frac{3}{2}\lim_{t\to 0^+}(5^{2/3} - t^{2/3}) = \frac{3}{2}5^{2/3}$$

F. Split at discontinuity at $c = 2$. Use $u = w - 2$ to show one integral diverges. Thus the original integral diverges.

DIVERGES, TYPE 2

$$\int_0^5 \frac{w}{w-2}\,dw = \int_0^2 \frac{w}{w-2}\,dw + \int_2^5 \frac{w}{w-2}\,dw$$

and

$$\int_0^2 \frac{w}{w-2}\,dw = \int_{-2}^0 \frac{u+2}{u}\,du = 2 + 2\lim_{t\to 0^-}\int_{-2}^t \frac{du}{u}$$
$$= 2 + 2\lim_{t\to 0^-}\Big[\ln|u|\Big]_{-2}^t = 2 + 2\lim_{t\to 0^-}(\ln(-t) - \ln 2) = -\infty$$

G. TYPE 1

$$\int_{-\infty}^0 2^r\,dr = \lim_{t\to-\infty}\left[\frac{2^r}{\ln 2}\right]_t^0 = \frac{1}{\ln 2}\lim_{t\to-\infty} 1 - 2^t = \frac{1}{\ln 2}(1-0) = \frac{1}{\ln 2}$$

H. Use $u = \sqrt{y}$ and then integrate by parts with $w = u$ and $dv = e^{-u}\,du$: TYPE 1

$$\int_0^\infty e^{-\sqrt{y}}\,dy = \lim_{t\to\infty}\int_0^t e^{-\sqrt{y}}\,dy = \lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-u}(2u)\,du = 2\lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u}\,du$$

$$= 2\lim_{t\to\infty}\left(\left.-ue^{-u}\right]_0^{\sqrt{t}} + \int_0^{\sqrt{t}} e^{-u}\,du\right) = 2\lim_{t\to\infty}\left(-\sqrt{t}e^{-\sqrt{t}} + \left[-e^{-u}\right]_0^{\sqrt{t}}\right)$$

$$= 2\lim_{t\to\infty}\left(-\sqrt{t}e^{-\sqrt{t}} - e^{-\sqrt{t}} + 1\right) = 2(0+0+1) = 2$$

I. Split at discontinuity $x = 1$ and compute each improper integral using $u = x - 1$: TYPE 2

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = -\frac{3}{2} + 6 = \frac{9}{2}$$

because

$$\int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_{-1}^0 u^{-1/3}\,du = \lim_{t\to 0^-}\left[\frac{3}{2}u^{2/3}\right]_{-1}^t = \frac{3}{2}\lim_{t\to 0^-}\left(t^{2/3} - 1\right) = -\frac{3}{2}$$

and

$$\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^8 u^{-1/3}\,du = \lim_{t\to 0^+}\left[\frac{3}{2}u^{2/3}\right]_t^8 = \frac{3}{2}\lim_{t\to 0^+}\left(8^{2/3} - t^{2/3}\right) = \frac{3}{2}(4-0) = 6$$

J. Use trig. identity but discover that one term dominates. DIVERGES, TYPE 1

$$\int_0^\infty \sin^2\alpha\,d\alpha = \frac{1}{2}\int_0^\infty 1 - \cos 2\alpha\,d\alpha = \frac{1}{2}\lim_{t\to\infty}\int_0^t 1 - \cos 2\alpha\,d\alpha$$

$$= \frac{1}{2}\lim_{t\to\infty}\left[\alpha - \frac{\sin 2\alpha}{2}\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left[t - \frac{\sin 2t}{2} - 0\right] = \infty$$
