Math 252 Calculus II (Bueler) 2 February 2018

Clean-up notes: Integrals of powers of tan and sec

These notes organize what was un-organized in lecture on Thursday 1 February.

The material in section 7.2 of Stewart ($8e$ edition) includes indefinite integrals of the form

$$\int \tan^m x \, \sec^n x \, dx$$

where m and n are non-negative integers (whole numbers). The table below organizes how to do all such integrals. Several times we use this trigonometric identity:

$$(*) \qquad 1 + \tan^2\theta = \sec^2\theta.$$

Also note that once one gets to the integral of a polynomial in u one can be confident of success. (So I stop with "...")

integral	method
$\int \tan x \, dx$	Easy using $u = \cos x$: $$\int \tan x \, dx = \int \frac{\sin x}{\cos x}\, dx = -\ln\lvert\cos x\rvert + C$$
$\int \sec x \, dx$	Tricky. Get: $$\int \sec x \, dx = \ln\lvert\sec x + \tan x\rvert + C$$
$\int \tan^{\text{any}} x \, dx$ (any ≥ 2)	Reducible using $(*)$ and $u = \tan x$: $$\int \tan^m x \, dx = \int \tan^{m-2} x \sec^2 x \, dx - \int \tan^{m-2}\, dx = \frac{1}{m-1}(\tan x)^{m-1} - \int \tan^{m-2}\, dx$$
$\int \tan^{\text{any}} x \, \sec^{\text{even}} x \, dx$ (even ≥ 2)	Pull out $\sec^2 x$ for du and use $(*)$ on remaining $\sec x$, then $u = \tan x$: $$\int \tan^m x \, \sec^{2k} x \, dx = \int \tan^m x \sec^{2k-2} x \, \sec^2 x \, dx = \int \tan^m x \, (1+\tan^2 x)^{k-1} \sec^2 x \, dx = \int u^m \, (1+u^2)^{k-1}\, du = \dots$$
$\int \tan^{\text{odd}} x \, \sec^{\text{any}} x \, dx$ (any ≥ 1)	Pull out $\sec x \tan x$ for du and use $(*)$ on remaining $\tan x$, then $u = \sec x$: $$\int \tan^{2k+1} x \, \sec^{m+1} x \, dx = \int \tan^{2k} x \sec^m x \, \sec x \tan x \, dx = \int (\sec^2 x - 1)^k \sec^m x \, \sec x \tan x \, dx = \int (u^2-1)^k \, u^m \, du = \dots$$

The next table has tricky/hard cases which will **not be on quizzes and exams**.

integral	**method**
$\int \sec^{\text{odd}} x\,dx$ (odd ≥ 3)	Separate one $\sec x$. Then use (∗) on remaining $\sec x$. Then use known integral for $\int \sec x\,dx$ and integration-by-parts on other integral(s) with $dv = \sec x \tan x\,dx$. Get "$I = [\text{known}] - cI$" type equations.
$\int \tan^{\text{even}} x\ \sec^{\text{odd}} x\,dx$ (even ≥ 2)	Use (∗) to replace all $\tan x$ with $\sec x$. Get integrals of previous type.