SOLUTIONS

Math F251: Section 3.5 Worksheet

1. Find dy/dx by implicit differentiation.

$$y\cos x = x^2 + y^2$$

$$\frac{dy}{dx}\cos x + y(-\sin x) = 2x + 2y\frac{dy}{dx}$$

product rule on left, chain rule on right

$$\frac{dy}{dx}(\cos x - 2y) = 2x + y\sin x$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y\sin x}{\cos x - 2y}}$$

2. Consider the equation

$$\sqrt{x} + \sqrt{y} = 1 \qquad (*)$$

(a) Find y' by implicit differentiation.

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

$$\boxed{y' = \frac{-\frac{1}{2}x^{-1/2}}{\frac{1}{2}y^{-1/2}} = -\frac{\sqrt{y}}{\sqrt{x}}}$$

(b) Solve $(*)$ explicitly for y and differentiate to get y' in terms of x.

$$\sqrt{y} = 1 - \sqrt{x}$$

$$y = (1-\sqrt{x})^2$$

$$\boxed{y' = 2(1-\sqrt{x})\left(-\frac{1}{2}x^{-1/2}\right) = -\frac{1-\sqrt{x}}{\sqrt{x}} = -x^{-1/2} + 1}$$

(c) Check that your solutions in (a) and (b) are consistent.

insert $y = (1-\sqrt{x})^2$ into (a) result:

$$y' = -\frac{\sqrt{(1-\sqrt{x})^2}}{\sqrt{x}} = -\frac{1-\sqrt{x}}{\sqrt{x}}$$

$$= -x^{-1/2} + 1 \quad \checkmark$$

3. (*A §3.4 question.*) For what values of r does the function $y = e^{rt}$ satisfy the differential equation $y'' - 4y' + y = 0$?

$$y' = re^{rt}$$
$$y'' = r^2 e^{rt}$$

$$y'' - 4y' + y = r^2 e^{rt} - 4re^{rt} + e^{rt} = 0$$

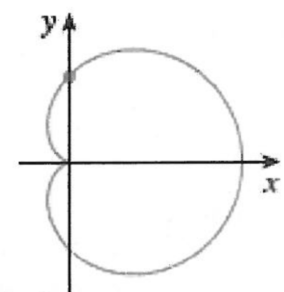
$$(e^{rt} \neq 0)$$

$$r^2 - 4r + 1 = 0$$

$$\boxed{r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}}$$

4. For the "cardiod" shown, with the equation and point given, find an equation of the tangent line.

$x^2+y^2 = (2x^2+2y^2-x)^2, \qquad (0, \frac{1}{2})$



$$2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)$$
$$= 2(2x^2+2y^2-x)(4x-1)$$
$$+ 8y(2x^2+2y^2-x)y'$$

$$y'(2y - 8y(2x^2 + 2y^2 - x)) = -2x + 2(2x^2 + 2y^2 - x)(4x-1)$$

$$y' = \frac{-2x + 2(2x^2+2y^2-x)(4x-1)}{2y - 8y(2x^2+2y^2-x)}$$

$$m = y'\Big|_{(0,\frac{1}{2})} = \frac{0 + 2(0+\frac{1}{2}-0)(-1)}{1 - 4(0+\frac{1}{2}-0)} = \frac{-1}{-1} = 1 \qquad \therefore \boxed{y - \frac{1}{2} = 1(x-0)}$$

5. If $xy + e^y = e$, find the value of y'' at the point where $x = 0$.

$$1 \cdot y + x \cdot y' + e^y y' = 0 \qquad @x=0: \quad 0 + e^y = e$$
$$\therefore y = 1$$

$$y' = \frac{-y}{x+e^y} \qquad \therefore y' = \frac{-1}{0+e} = -\frac{1}{e}$$

$$y'' = \frac{-y'(x+e^y) + y(1+e^y y')}{(x+e^y)^2}$$

$$\boxed{y'' = \frac{+\frac{1}{e}(0+e) + 1(1+e(\frac{-1}{e}))}{(0+e)^2} = \frac{1+1-1}{e^2} = \frac{1}{e^2}}$$