Newton's method solves a problem

$$f(x) = 0$$

by starting with an initial estimate x_0. It gets a new estimate from the old estimate by finding where the linearization $L(x) \approx f(x)$ crosses the x-axis. (*Write down the linearization $L(x)$ at $a = x_0$. Solve $L(x) = 0$ for x.*) So it uses this formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example 1. Suppose you want to solve the equation

$$3x^4 - x^3 + 6x^2 + 4x - 2 = 0.$$

I do not see how to factor this by hand. (*Do you?*) If we name the left side of the equation "$f(x)$" and if we go looking for a root of $f(x)$ then we might notice $f(0) = -2$ and $f(1) = 10$. Because f is continuous, by the intermediate value theorem there is a root on $(0, 1)$. Also, the derivative is an easy calculation: $f'(x) = 12x^3 - 3x^2 + 12 + 4$.

Newton's method with initial estimate $x_0 = 1/2$ gives the next value by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.354651162790698$$

(*It is nice to immediately use a* serious *calculator, namely one with a big screen and the ability to run a loop. Thus I used* MATLAB *on my laptop.*) The next four iterations gave consistent numbers which strongly suggest $x = 1/3$ is an exact root:

$$\begin{aligned}
x_0 &= 0.500000000000000\\
x_1 &= \underline{0.3}54651162790698\\
x_2 &= \underline{0.333}718551391767\\
x_3 &= \underline{0.333333}461355651\\
x_4 &= \underline{0.3333333333333}47\\
x_5 &= \underline{0.333333333333333}
\end{aligned}$$

Notice how the number of correct digits doubles from x_1 to x_4. It stops doubling only because the computer only stores about 16 decimal digits. In fact we have discovered how to factor the function: $f(x) = (x - \frac{1}{3})(3x^3 + 6x + 6)$.

Example 2. Suppose we want to find the maximum of $g(x) = x^2 - 2\sin x$ on the interval $[0, 1]$. We take the derivative and set it to zero to find the critical numbers:

$$2x - 2\cos x = 0$$

You can factor and remove the "2," but otherwise this is an equation I do not know how to solve by hand. So I try Newton's method. Let $f(x) = x - \cos x$ so $f'(x) = 1 + \sin(x)$. Newton's method starting with $x_0 = 0.5$ gives these five iterates, and the last two agree to all digits:

$$\begin{aligned}
x_0 &= 0.500000000000000\\
x_1 &= \underline{0.7}55222417105636\\
x_2 &= \underline{0.739}141666149879\\
x_3 &= \underline{0.739085133}920807\\
x_4 &= \underline{0.739085133215161}\\
x_5 &= \underline{0.739085133215161}
\end{aligned}$$

In fact $f(x_4) = 0$ to the accuracy of the computer. Again notice the approximate doubling of the number of correct digits at each Newton step.