# Solutions to Worksheet on “Areas under curves by limits of sums of rectangles”

**(a)** The drawing is omitted, but it shows a parabolic arc which hits the $x$-axis at $x = -2$ and $x = 2$. Thus the interval is $[a, b] = [-2, 2]$. The vertex (peak) of the parabola is at point $(0, 4)$. By putting a rectangle around the area we want, call it $A$, and by putting a triangle inside the area, we also see that

$$8 \le A \le 16.$$

This will help us know if we have a reasonable answer.

**(b)**

$$\Delta x = \frac{2-(-2)}{n} = \frac{4}{n}, \qquad x_i = a + i\Delta x = -2 + i\frac{4}{n}.$$

**(c)**

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x = \sum_{i=1}^{n} \left(4 - \left(-2 + (i-1)\frac{4}{n}\right)^2\right) \frac{4}{n}$$

**(d)**

$$\begin{aligned}
s(n) &= \frac{4}{n}\sum_{i=1}^{n} 4 - \left(-2 + (i-1)\frac{4}{n}\right)^2 \\
&= \frac{4}{n}\sum_{i=1}^{n} 4 - \left(4 - 4(i-1)\frac{4}{n} + (i-1)^2\frac{4^2}{n^2}\right) \\
&= \frac{4}{n}\sum_{i=1}^{n} \left(\frac{16}{n}(i-1) - \frac{16}{n^2}(i-1)^2\right) \\
&= \frac{64}{n^2}\sum_{i=1}^{n}(i-1) - \frac{64}{n^3}\sum_{i=1}^{n}(i-1)^2 \\
&= \frac{64}{n^2}\left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right) - \frac{64}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right)
\end{aligned}$$

**(e)** continuing using formulas for $\sum i^2, \sum i, \sum c$ printed on page 296:

$$\begin{aligned}
s(n) &= \frac{64}{n^2}\left(\frac{n(n+1)}{2} - n\right) - \frac{64}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n\right) \\
&= \frac{64}{n}\left(\frac{n+1}{2} - 1\right) - \frac{64}{n^2}\left(\frac{(n+1)(2n+1)}{6} - (n+1) + 1\right) \\
&= 32\frac{n+1}{n} - \frac{64}{n} - \frac{32}{3}\frac{(n+1)(2n+1)}{n^2} + 64\frac{n+1}{n^2} - \frac{64}{n^2}
\end{aligned}$$

**(f)** finally we can get the area by taking the limit:

$$\begin{aligned} A &= \lim_{n\to\infty} s(n) \\ &= \lim_{n\to\infty} 32\frac{n+1}{n} - \frac{64}{n} - \frac{32}{3}\frac{(n+1)(2n+1)}{n^2} + 64\frac{n+1}{n^2} - \frac{64}{n^2} \\ &= 32\lim_{n\to\infty}\frac{n+1}{n} - \lim_{n\to\infty}\frac{64}{n} - \frac{32}{3}\lim_{n\to\infty}\frac{(n+1)(2n+1)}{n^2} + 64\lim_{n\to\infty}\frac{n+1}{n^2} - \lim_{n\to\infty}\frac{64}{n^2} \\ &= 32\cdot 1 - 0 - \frac{32}{3}\cdot 2 + 0 - 0 \\ &= 32 - \frac{64}{3} \\ &= \frac{32}{3} \\ &= 10.6666666\cdots \end{aligned}$$

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Notes:

- The calculation would be easier if we used right endpoints because "$i$" would replace "$(i-1)$" in many expressions.
- Because $f(x) = 4 - x^2$ is an even function, we could find the area under $f(x)$ on the interval $[0, 2]$ and then double that to get our answer. This would be easier because
$$x_i = a + i\Delta x = 0 + i\frac{2}{n} = i\frac{2}{n}$$
in that case.
- Once we have the Fundamental Theorem of Calculus (section 5.4), and using the definite integral notation in section 5.3, this calculation is vastly quicker:

$$A = \int_{-2}^{2} 4 - x^2\,dx = \left[4x - \frac{x^3}{3}\right]_{x=-2}^{x=2} = \left(8 - \frac{(2)^3}{3}\right) - \left(-8 - \frac{(-2)^3}{3}\right) = 8 - \frac{8}{3} + 8 - \frac{8}{3} = \frac{32}{3}.$$